

* Evaluation of integrals by Laplace transforms:-

Step 1:- Comparing the given integral with the definition of Laplace transform and define

$$f(t) = \frac{1}{2} \left(\frac{1}{1+e^t} - \frac{1}{1+e^{-t}} \right) = \frac{1}{2} \left(\frac{e^{-t/2}}{1+e^{t/2}} - \frac{e^{t/2}}{1+e^{-t/2}} \right)$$

Step 2:- Find $L[f(t)] = \bar{f}(s)$

Step 3:- Substitute 's' value in $\bar{f}(s)$.

$$\begin{aligned} & \frac{1}{2} \left(\frac{1}{1+e^t} - \frac{1}{1+e^{-t}} \right) = \frac{1}{2} \left(\frac{e^{-t/2}}{1+e^{t/2}} - \frac{e^{t/2}}{1+e^{-t/2}} \right) \\ & = \frac{1}{2} \left(\frac{e^{-t/2}}{1+e^{t/2}} - \frac{e^{t/2}}{1+e^{-t/2}} \right) = \frac{1}{2} \left(\frac{e^{-t/2}}{1+e^{t/2}} - \frac{e^{t/2}}{1+e^{-t/2}} \right) \\ & = \frac{1}{2} \left(\frac{e^{-t/2}}{1+e^{t/2}} - \frac{e^{t/2}}{1+e^{-t/2}} \right) = \frac{1}{2} \left(\frac{e^{-t/2}}{1+e^{t/2}} - \frac{e^{t/2}}{1+e^{-t/2}} \right) \end{aligned}$$

Evaluate $\int_0^{\infty} e^{-3t} \sin t \, dt$ using Laplace Transform.

$$\begin{aligned} \text{Given } \int_0^{\infty} e^{-3t} \sin t \, dt &= L[\sin t] \left[\because \int_0^{\infty} e^{-st} f(t) \, dt = \bar{f}(s) \right] \\ &= \frac{1}{s^2 + 1} \\ &= \frac{1}{3^2 + 1} = \frac{1}{10} \end{aligned}$$

Evaluate $\int_0^{\infty} t e^{-2t} \cos t \, dt$

$$\begin{aligned} \text{Given } \int_0^{\infty} t e^{-2t} \cos t \, dt &= \int_0^{\infty} e^{-2t} (t \cos t) \, dt \\ &= L[t \cos t] \left[\because \int_0^{\infty} e^{-st} f(t) \, dt = L[f(t)] \right] \\ &= -\frac{d}{ds} [L[\cos t]] \left[\because L[t f(t)] = -\frac{d}{ds} (\bar{f}(s)) \right] \\ &= -\frac{d}{ds} \left(\frac{s}{s^2 + 1} \right) = -\frac{d}{ds} \left[L[\cos t] \right] \\ &= -\left[\frac{(s^2 + 1)(1) - s(2s)}{(s^2 + 1)^2} \right] \\ &= -\left[\frac{1 - s^2}{(s^2 + 1)^2} \right] = \frac{s^2 - 1}{(s^2 + 1)^2} = \frac{2^2 - 1}{(2^2 + 1)^2} \\ &= \frac{3}{25} \end{aligned}$$

Evaluate $\int_0^{\infty} t e^{-2t} \cos t \, dt = \int_0^{\infty} \frac{\sin t}{t} \, dt$ using L.T.

The given integral can be written as $\int_0^{\infty} e^{-st} \frac{\sin t}{t} \, dt$.
This is of the form $\int_0^{\infty} e^{-st} f(t) \, dt$ where $f(t) = \frac{\sin t}{t}$
and $s=0$.

By Laplace transform formula $\int_0^{\infty} e^{-st} f(t) \, dt = L[f(t)]$

$$\int_0^{\infty} e^{-st} \frac{\sin t}{t} \, dt = L\left[\frac{\sin t}{t}\right]$$

$$= \int_0^{\infty} L(\sin t) \, ds$$

$$= \int_0^{\infty} \frac{1}{s^2 + 1} \, ds = (\tan^{-1} s)_0^{\infty}$$

$$= \tan^{-1} \infty - \tan^{-1} s$$

$$= \tan^{-1}(\tan \pi/2) - \tan^{-1} s$$

$$= \pi/2 - \tan^{-1} s = \cot^{-1} s$$

$$= \cot^{-1} 0 \text{ As } s \rightarrow 0$$

$$= \cot^{-1}(\cot \pi/2) = \pi/2.$$

4. Evaluate $\int_0^{\infty} \frac{e^{-t} - e^{-3t}}{t} dt$ using Laplace Transforms.

The given integral can be written as $\int_0^{\infty} e^{-st} \left(\frac{e^{-t} - e^{-3t}}{t} \right) dt$

This is of the form $\int_0^{\infty} e^{-st} f(t) dt$ with $f(t)$

$$f(t) = \frac{e^{-t} - e^{-3t}}{t} \text{ and } s=0.$$

By Laplace transform formula $\int_0^{\infty} e^{-st} f(t) dt = L[f(t)]$

$$\int_0^{\infty} e^{-st} \left(\frac{e^{-t} - e^{-3t}}{t} \right) dt = L\left[\frac{e^{-t} - e^{-3t}}{t}\right]$$

$$= \int_0^{\infty} [e^{-t} - e^{-3t}] ds$$

$$= \int_0^{\infty} [L(e^{-t}) - L(e^{-3t})] ds$$

$$= \int_0^{\infty} \left[\frac{1}{s+1} - \frac{1}{s+3} \right] ds$$

$$= \int_0^{\infty} (\log s+1 - \log s+3) ds = \left(\log \frac{s+1}{s+3} \right)_s$$

$$= \left(\frac{\log s(1 + \frac{1}{s})}{s(1 + \frac{3}{s})} \right)_s = \log \frac{1+0}{1+0} - \log \frac{1 + \frac{1}{s}}{1 + \frac{3}{s}}$$

$$= \log 1 - \log \frac{s+1}{s+3}$$

$$= 0 - \log \left(\frac{s+1}{s+3} \right) = \log \frac{s+3}{s+1}$$

$$= \log \frac{0+3}{0+1} = \log 3.$$

As $s \rightarrow 0$.

5. Evaluate $\int_0^{\infty} e^{-2t} \sin^3 t \, dt$ ③ $\sin 3t = 3 \sin t - 4 \sin^3 t$
 $\sin^3 t = \frac{3 \sin t - \sin 3t}{4}$
 The given integral is of the form $\int_0^{\infty} e^{-st} f(t) \, dt$
 with $f(t) = \sin^3 t$ and $s=2$.

$$f(t) = \frac{3 \sin t - \sin 3t}{4}$$

By Laplace transform formula,

$$\int_0^{\infty} e^{-st} f(t) \, dt = L[f(t)]$$

$$= L\left[\frac{3 \sin t - \sin 3t}{4}\right] = \frac{3}{4} L[\sin t] - \frac{1}{4} L[\sin 3t]$$

$$= \frac{3}{4} \left(\frac{1}{s^2+1}\right) - \frac{1}{4} \left(\frac{3}{s^2+9}\right) = \frac{3}{4} \left(\frac{1}{s^2+1} - \frac{1}{s^2+9}\right)$$

$$= \frac{3}{4} \left(\frac{s^2+9-s^2-1}{(s^2+1)(s^2+9)}\right) = \frac{3}{4} \left(\frac{8}{(s^2+1)(s^2+9)}\right) = \frac{6}{(s^2+1)(s^2+9)}$$

$$= \frac{6}{(2^2+1)(2^2+9)} = \frac{6}{5 \times 13} = \frac{6}{65}$$

(As $s=2$)

Laplace transform of derivatives:-

If $L[f(t)] = \bar{f}(s)$, then $L[f'(t)] = s\bar{f}(s) - f(0)$, where
 $\bar{f}(s) = L[f(t)]$

Note (i) $L[f''(t)] = s^2 \bar{f}(s) - sf(0) - f'(0)$

(ii) $L[f'''(t)] = s^3 \bar{f}(s) - s^2 f(0) - sf'(0) - f''(0)$

In general $L[f^{(n)}(t)] = s^n \bar{f}(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \dots - f^{(n-1)}(0)$

If $L[\sin \sqrt{t}] = \frac{\sqrt{\pi}}{2s^{3/2}} e^{-1/4s}$ find $L\left[\frac{\cos \sqrt{t}}{\sqrt{t}}\right]$

Let $f(t) = \sin \sqrt{t}$ then $L[f(t)] = \frac{\sqrt{\pi}}{2s^{3/2}} e^{-1/4s} = \bar{f}(s)$

$f'(t) = \cos \sqrt{t} \cdot \frac{1}{2\sqrt{t}}$ Also $f(0) = \sin 0 = 0$

We know that $L[f'(t)] = s\bar{f}(s) - f(0)$

$$\therefore L\left[\frac{\cos \sqrt{t}}{2\sqrt{t}}\right] = s \frac{\sqrt{\pi}}{2s^{3/2}} e^{-1/4s} - 0$$

$$\frac{1}{2} L \left[\frac{\cos \sqrt{t}}{\sqrt{t}} \right] = \frac{\sqrt{\pi}}{2s^{3/2-1}} e^{-1/4s} \quad (4)$$

$$L \left[\frac{\cos \sqrt{t}}{\sqrt{t}} \right] = \frac{\sqrt{\pi}}{s^{1/2}} e^{-1/4s} = \sqrt{\frac{\pi}{s}} e^{-1/4s}$$

2. Given $L[t \sin at] = \frac{2as}{(s^2+a^2)^2}$ evaluate (i) $L[at \cos at + \sin at]$

(ii) $L[2a \cos at - at \sin at]$

Let $f(t) = t \sin at$ & $L[f(t)] = \frac{2as}{(s^2+a^2)^2} = F(s) \quad \text{--- (1)}$

then $f'(t) = t a \cos at + \sin at$ (i)
 $= at \cos at + \sin at$

Also $f''(t) = a[t(-a) \sin at + \cos at]$ (ii) $+ a \cos at$

$= -a^2 t \sin at + a \cos at + a \cos at$
 $= 2a \cos at - a^2 t \sin at$

Also $f(0) = 0$, $f'(0) = a(0) + 0 = 0$

(i) We know that $L[f'(t)] = sF(s) - f(0)$
 $= s \left(\frac{2as}{(s^2+a^2)^2} \right) - 0 = \frac{2as^2}{(s^2+a^2)^2}$

(ii) $L[f''(t)] = s^2 F(s) - sf'(0) - f'(0)$

$L[2a \cos at - a^2 t \sin at] = s^2 \left(\frac{2as}{(s^2+a^2)^2} \right) - s(0) - 0 = \frac{2as^3}{(s^2+a^2)^2}$

$a L[2 \cos at - at \sin at] = \frac{2as^3}{(s^2+a^2)^2}$

$L[2 \cos at - at \sin at] = \frac{2s^3}{(s^2+a^2)^2}$

3. Using the theorem on Laplace Transform of derivative

find $L[\cos at]$

Let $f(t) = \cos at$, $f'(t) = -a \sin at$, $f''(t) = -a^2 \cos at$

Also $f(0) = 1$

$f'(0) = -a \sin 0 = 0$

Using theorem of Laplace Transform of derivatives

$$L[f''(t)] = s^2 \bar{f}(s) - s f(0) - f'(0) \quad (5)$$

$$L[-a^2 \cos at] = s^2 \left(\frac{2as}{s^2 + a^2} \right) L[\cos at] - s(1) - 0$$

$$-a^2 L[\cos at] = s^2 L[\cos at] - s$$

$$s^2 L[\cos at] + a^2 L[\cos at] = s$$

$$(s^2 + a^2) L[\cos at] = s \Rightarrow L[\cos at] = \frac{s}{s^2 + a^2}$$

$$4. L\left[2\sqrt{\frac{t}{\pi}}\right] = \frac{1}{s^{3/2}} \text{ find } L\left[\frac{1}{\sqrt{\pi t}}\right]$$

$$\text{Let } f(t) = 2\sqrt{\frac{t}{\pi}} \quad f'(t) = \frac{2}{\sqrt{\pi}} \left(\frac{1}{2\sqrt{t}} \right) = \frac{1}{\sqrt{\pi t}}$$

$$f(0) = 0 \quad f'(0) = \infty$$

Using theorem of Laplace transform of derivatives

$$L[f'(t)] = s \bar{f}(s) - f(0)$$

$$= s \left(\frac{1}{s^{3/2}} \right) - 0 = \frac{1}{\sqrt{s}}$$

5. $L[e^{at}]$ Using L.T of derivatives

$$\text{Let } f(t) = e^{at} \quad f'(t) = ae^{at} \quad f''(t) = a^2 e^{at} \quad \text{Also } f(0) = 1 \quad f'(0) = a$$

$$\text{Using L.T of derivatives, } L[f''(t)] = s^2 \bar{f}(s) - s f(0) - f'(0)$$

$$L[a^2 e^{at}] = s^2 L[e^{at}] - s(1) - a \Rightarrow a^2 L[e^{at}] = s^2 L[e^{at}] - s - a$$

$$\Rightarrow (a^2 - s^2) L[e^{at}] = -(s+a) \Rightarrow (s+a)(s-a) L[e^{at}] = -(s+a)$$

$$\Rightarrow (a^2 - s^2) L[e^{at}] = -(s+a) \Rightarrow (s+a)(s-a) L[e^{at}] = -(s+a) \Rightarrow L[e^{at}] = \frac{1}{s-a}$$

* Laplace Transform of Integrals:-

$$\text{If } L[f(t)] = \bar{f}(s) \text{ then (i) } L\left[\int_0^t f(u) du\right] = \frac{1}{s} \bar{f}(s)$$

$$(ii) L\left[\int_0^t \int_0^u f(u) du du\right] = \frac{1}{s^2} \bar{f}(s)$$

$$\text{Integral } L\left[\int_0^t \int_0^u \dots \int_0^t f(u) du du \dots du\right]_{n \text{ times}} = \frac{1}{s^n} \bar{f}(s)$$

1. Find $L\left[\int_0^t e^{-t} \cos t \, dt\right]$

(6)

Let $f(t) = e^{-t} \cos t$ then

$$L[f(t)] = L[e^{-t} \cos t]$$

$$= L[\cos t]_{s \rightarrow s+1} = \left[\frac{s}{s^2+1} \right]_{s \rightarrow s+1}$$

$$= \frac{s+1}{(s+1)^2+1} = \frac{s+1}{s^2+2s+2} = F(s)$$

Using theorem on L.T of integrals.

$$L\left[\int_0^t f(u) \, du\right] = \frac{1}{s} F(s)$$

$$L\left[\int_0^t e^{-t} \cos t \, dt\right] = \frac{1}{s} \left(\frac{s+1}{s^2+2s+2} \right)$$

2. Find $L\left[\int_0^t t \sin 3t \, dt\right]$

Let $f(t) = t \sin 3t$ then

$$L[f(t)] = L[t \sin 3t]$$

$$= -\frac{d}{ds} L(\sin 3t) = -\frac{d}{ds} \left(\frac{3}{s^2+9} \right)$$

$$= -3 \left[-\frac{1}{(s^2+9)^2} (2s) \right] = \frac{6s}{(s^2+9)^2} = F(s)$$

Using theorem on L.T of integrals

$$L\left[\int_0^t f(u) \, du\right] = \frac{1}{s} F(s)$$

$$L\left[\int_0^t t \sin 3t \, dt\right] = \frac{1}{s} \left(\frac{6s}{(s^2+9)^2} \right) = \frac{6}{(s^2+9)^2}$$

3. $L\left[\int_0^t e^{-t} \frac{\sin t}{t} \, dt\right]$

Let $f(t) = e^{-t} \frac{\sin t}{t}$

$$L\left[\frac{\sin t}{t}\right] = \int_s^\infty L(\sin t) \, ds = \int_s^\infty \frac{1}{s^2+1} \, ds$$

$$= \left[\tan^{-1}(s) \right]_s^\infty$$

$$= \left[\left[-\tan^{-1} s - \tan^{-1} s \right]_s^{\infty} \right]_{s \rightarrow s+1} \quad (7)$$

$$= \left[\frac{\pi}{2} - \tan^{-1} s \right]_{s \rightarrow s+1}$$

$$= \left[\cot^{-1} s \right]_{s \rightarrow s+1} = \cot^{-1}(s+1)$$

Using theorem on L.T of integrals

$$L \left[\int_0^t f(u) du \right] = \frac{1}{s} \bar{f}(s)$$

$$L \left[\int_0^t e^{-t} \frac{\sin t}{t} dt \right] = \frac{1}{s} \cot^{-1}(s+1)$$

4. Find $L \left[e^{-t} \int_0^t \frac{\sin t}{t} dt \right]$

Let $g(t) = \int_0^t \frac{\sin t}{t} dt$ then

$$L[g(t)] = L \left[\int_0^t \frac{\sin t}{t} dt \right]$$

Let $f(t) = \frac{\sin t}{t}$ then $L[f(t)] = \int_s^{\infty} L[\sin t] ds$

$$= \int_s^{\infty} \frac{1}{s^2+1} ds$$

$$= \left(\tan^{-1} s \right)_s^{\infty} = \tan^{-1} \infty - \tan^{-1} s$$

$$= \frac{\pi}{2} - \tan^{-1} s = \cot^{-1} s = \bar{f}(s)$$

Using theorem of L.T of integrals

$$L[g(t)] = L \left[\int_0^t \frac{\sin t}{t} dt \right] = \frac{1}{s} \bar{f}(s) = \frac{1}{s} \cot^{-1} s = \frac{1}{s} \cot^{-1} s$$

$$\therefore L \left[e^{-t} \int_0^t \frac{\sin t}{t} dt \right] = \left[L \left[\int_0^t \frac{\sin t}{t} dt \right] \right]_{s \rightarrow s+1} \quad (\text{By F.S.T})$$

$$= \left[\frac{1}{s} \cot^{-1}(s) \right]_{s \rightarrow s+1}$$

$$= \frac{\cot^{-1}(s+1)}{s+1}$$

* II Shifting Theorem :-

$$\text{If } L[f(t)] = \bar{f}(s) \text{ and } g(t) = f(t-a) \text{ if } t > a \\ = 0 \text{ if } t < a$$

$$\text{then } L[g(t)] = e^{-as} \bar{f}(s) \\ = e^{-as} L[f(t)]$$

1. Find the Laplace transform of $g(t) = \sin\left(t - \frac{2\pi}{3}\right)$

Sol:- Comparing the given function with $f(t-a)$

$$f(t) = \sin t, a = \frac{2\pi}{3}$$

$$L[f(t)] = L[\sin t] = \frac{1}{s^2+1} = \bar{f}(s) \quad \text{--- (1)}$$

By second shifting theorem

$$L[g(t)] = e^{-as} \bar{f}(s) \\ = e^{-2\pi/3 s} \left(\frac{1}{s^2+1} \right) \quad [\text{From (1)}]$$

2. Find L.T of $g(t) = \cos\left(t - \frac{2\pi}{3}\right)$ if $t > \frac{2\pi}{3}$
 $= 0$ if $t < \frac{2\pi}{3}$

Comparing the given function with $f(t-a)$

$$f(t) = \cos t, a = \frac{2\pi}{3}$$

$$L[f(t)] = L[\cos t] = \frac{s}{s^2+1} = \bar{f}(s)$$

By second shifting theorem

$$L[g(t)] = e^{-as} \bar{f}(s) \\ = e^{-2\pi/3 s} \left(\frac{s}{s^2+1} \right) \quad [\text{from (1)}]$$

3 Find L.T of $g(t) = \cos\left(t - \frac{\pi}{3}\right)$ if $t > \pi/3$
 $= 0$ if $t < \pi/3$

Comparing given function with $f(t-a)$

$$f(t) = \cos t, a = \pi/3$$

$$L[f(t)] = L[\cos t] = \frac{s}{s^2+1} = \bar{f}(s)$$

By second shifting theorem

$$L[g(t)] = e^{-as} \bar{f}(s)$$

$$= e^{-\pi/3 s} \left(\frac{s}{s^2+1} \right)$$

* Unit step Function (or) Heavisides Unit Function

* Define unit step function and find its Laplace transform.

Def:- The unit step function $u(t-a)$ is defined

$$\text{as } u(t-a) \text{ or } H(t-a) = \begin{cases} 1, & \text{if } t \geq a \\ 0, & \text{if } t < a \end{cases} \quad [H(t-a)]$$

To find $L[u(t-a)]$

$$L[u(t-a)] = \int_0^{\infty} e^{-st} u(t-a) dt$$

$$= \left[\int_0^a e^{-st} (0) dt + \int_a^{\infty} e^{-st} (1) dt \right]$$

$$= \int_a^{\infty} e^{-st} dt$$

$$= \left[\frac{e^{-st}}{-s} \right]_a^{\infty} = -\frac{1}{s} [e^{-\infty} - e^{-as}]$$

$$= -\frac{1}{s} (0 - e^{-as})$$

$$\boxed{L[u(t-a)] = \frac{e^{-as}}{s}}$$

* Another form of second shifting Theorem:-

$$\text{If } L[f(t)] = \bar{f}(s) \text{ then } L[f(t-a)u(t-a)] = e^{-as} \bar{f}(s)$$

$$\text{where } u(t-a) = \begin{cases} 1 & \text{if } t \geq a \\ 0 & \text{if } t < a \end{cases}$$

1. Find the L.T of $(t-2)^3 u(t-2)$

$$L[(t-2)^3 u(t-2)] \text{ Comparing with } L[f(t-a)u(t-a)]$$

$$\text{we get } f(t) = t^3, a = 2$$

$$L[f(t)] = L[t^3] = \frac{3!}{s^{3+1}} = \frac{6}{s^4} = \bar{f}(s) \quad \text{--- (1)}$$

By Second shifting theorem

$$L[f(t-a)u(t-a)] = e^{-as} \bar{f}(s) \\ = e^{-2s} \frac{6}{s^4}$$

$$2. L[e^{-3t}u(t-2)]$$

$$= L[e^{-3(t-2)-6}u(t-2)]$$

$$= L[e^{-3(t-2)}e^{-6}u(t-2)]$$

$$= e^{-6} L[e^{-3(t-2)}u(t-2)] \quad \text{--- (1)}$$

$$\text{Let } f(t) = e^{-3t} \quad a = 2$$

$$\text{Then } L[f(t)] = L[e^{-3t}] = \frac{1}{s+3} = \bar{f}(s)$$

$$\text{By S.S.T, } L[f(t-a)u(t-a)] = e^{-as} \bar{f}(s)$$

$$= e^{-2s} \frac{1}{s+3} \quad \text{--- (2)}$$

Substitute (2) in (1) we get

$$L[e^{-3t}u(t-2)] = \frac{e^{-6}e^{-2s}}{s+3} = \frac{e^{-2s-6}}{s+3} \\ = \frac{e^{-2(s+3)}}{s+3}$$

* Laplace Transform of Periodic functions.

periodic function:- A function $f(t)$ is said to be periodic if and only if $f(t+T) = f(t)$ and $T > 0$, where T is the least positive value 'T' is known as period of $f(t)$.

Ex:- (i) $\sin t$ and $\cos t$ are periodic functions with period 2π .

(ii) $\tan t$ and $\cot t$ are periodic functions with period π .

If $f(t)$ is a periodic function of period T then $L[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t) dt$

1. Find $L[f(t)]$, where $f(t)$ is a periodic function of period 2π and it is given by $f(t) = \sin t$, $0 < t < \pi$
 $= 0$, $\pi < t < 2\pi$

Sol:- W.K.T $L[f(t)]$: If $f(t)$ is a periodic function of period T then $L[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t) dt$

Given $f(t) = \sin t$ $0 < t < \pi$
 $= 0$ $\pi < t < 2\pi$

Since $f(t)$ is a periodic function with period 2π i.e., $T = 2\pi$

$$\begin{aligned} \therefore L[f(t)] &= \frac{1}{1-e^{-s2\pi}} \int_0^{2\pi} e^{-st} f(t) dt \\ &= \frac{1}{1-e^{-s2\pi}} \int_0^{\pi} e^{-st} \sin t dt + \int_{\pi}^{2\pi} e^{-st} (0) dt \\ &= \frac{1}{1-e^{-s2\pi}} \int_0^{\pi} e^{-st} \sin t dt \end{aligned}$$

$\int e^{ax} \sin bx dx = \frac{e^{ax}}{a^2+b^2} (a \sin bx - b \cos bx)$

$$\begin{aligned}
 (1b) &= \frac{1}{1-e^{-2\pi s}} \left(\int_0^{\pi} e^{-st} \sin t \, dt \right) \\
 &= \frac{1}{1-e^{-2\pi s}} \left[\frac{e^{-st}}{(-s)^2+1^2} (-s \sin t - \cos t) \right]_0^{\pi} \\
 &= \frac{1}{1-e^{-2\pi s}} \left[\frac{e^{-st}}{s^2+1} (-s \sin t - \cos t) \right]_0^{\pi} \\
 &= \frac{1}{1-e^{-2\pi s}} \left[\frac{e^{-\pi s}}{s^2+1} (-s \sin \pi - \cos \pi) - \frac{e^{-0}}{s^2+1} (-s \sin 0 - \cos 0) \right] \\
 &= \frac{1}{1-e^{-2\pi s}} \left[\frac{e^{-\pi s}}{s^2+1} (0 - (-1)) - \frac{1}{s^2+1} (-s(0) - 1) \right] \\
 &= \frac{1}{1-e^{-2\pi s}} \left[\frac{e^{-\pi s}}{s^2+1} + \frac{1}{s^2+1} \right] = \frac{1}{s^2+1} \frac{(e^{-\pi s} + 1)}{(1-e^{-2\pi s})} \\
 &= \frac{1}{s^2+1} \frac{(e^{-\pi s} + 1)}{(1+e^{-\pi s})(1-e^{-\pi s})} \\
 &= \frac{1}{(s^2+1)(1-e^{-\pi s})}
 \end{aligned}$$

2. If $f(t) = 1, 0 \leq t < 1$
 $= -1, 1 \leq t < 2$ is a periodic function
 with period 2. Find its Laplace Transform.

Sol:- W.K.T $f(t)$ is a periodic function of period T

$$T \text{ then } L[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t) \, dt$$

$$\text{Given } f(t) = 1 \quad 0 \leq t < 1$$

$$= -1 \quad 1 \leq t < 2$$

Since $f(t)$ is periodic function of period 2 $T=2$

$$\therefore L[f(t)] = \frac{1}{1-e^{-2s}} \int_0^2 e^{-st} f(t) \, dt$$

$$= \frac{1}{1-e^{-2s}} \left[\int_0^1 e^{-st} f(t) \, dt + \int_1^2 e^{-st} f(t) \, dt \right]$$

$$= \frac{1}{1-e^{-2s}} \left[\int_0^1 e^{-st} (1) dt + \int_1^2 e^{-st} (-1) dt \right]$$

$$= \frac{1}{1-e^{-2s}} \left[\left(\frac{e^{-st}}{-s} \right)_0^1 + \left(-\frac{e^{-st}}{s} \right)_1^2 \right]$$

$$= \frac{1}{1-e^{-2s}} \left[-\frac{1}{s} (e^{-s} - e^0) + \frac{1}{s} (e^{-2s} - e^{-s}) \right]$$

$$= \frac{1}{1-e^{-2s}} \left[-\frac{1}{s} e^{-s} + \frac{1}{s} + \frac{1}{s} e^{-2s} - \frac{1}{s} e^{-s} \right]$$

$$= \frac{1}{1-e^{-2s}} \left[\frac{1}{s} (e^{-2s} + 1) \right] \left\{ \frac{1}{s} - \frac{(1-e^{-2s})}{1-e^{-2s}} = -\frac{1}{s} \right\}$$

$$= \frac{1}{1-e^{-2s}} \left[\frac{1}{s} e^{-2s} + \frac{1}{s} - \frac{2}{s} e^{-s} \right]$$

$$= \frac{1}{1-e^{-2s}} \left(\frac{1}{s} \right) (e^{-2s} - 2e^{-s} + 1)$$

$$= \frac{1}{s} \left(\frac{1}{1-e^{-2s}} \right) (e^{-s} - 1)^2$$

$$= \frac{1}{s} \left(\frac{1}{(1-e^{-s})(1+e^{-s})} \right) (1-e^{-s})^2$$

$$= \frac{1-e^{-s}}{s(1+e^{-s})}$$

3. Find the L.T of the sawtoothed wave of period T given by $f(t) = \frac{t}{T}$ for $0 < t < T$

W.K.T $L[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t) dt$

Given $f(t) = \frac{t}{T}$ $0 < t < T$

$$\therefore L[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} \frac{t}{T} dt$$

$$= \frac{1}{1-e^{-sT}} \cdot \frac{1}{T} \int_0^T e^{-st} t dt$$

$$= \frac{1}{T(1-e^{-sT})} \left[t \left(\frac{e^{-st}}{-s} \right) - (1) \left(\frac{e^{-st}}{s} \right) \right]_0^T$$

$$= \frac{1}{T(1-e^{-sT})} \left[\left(-\frac{T e^{-sT}}{s} - \frac{e^{-sT}}{s} \right) - \left(0 - \frac{e^{-0}}{s} \right) \right]$$

$$= \frac{1}{T(1-e^{-sT})} \left[-\frac{T}{s} e^{-sT} - \frac{1}{s^2} e^{-sT} + \frac{1}{s^2} \right]$$

4. Find L.T of full wave rectifier $f(t) = E \sin \omega t$
 $0 < t < \frac{\pi}{\omega}$ having period $\frac{\pi}{\omega}$

$$\text{W.K.T } L[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{st} f(t) dt$$

Given $f(t) = E \sin \omega t$ $0 < t < \frac{\pi}{\omega}$

$$\therefore L[f(t)] = \frac{1}{1-e^{-\pi s/\omega}} \int_0^{\pi/\omega} e^{-st} (E \sin \omega t) dt$$

$$= \frac{E}{1-e^{-\pi s/\omega}} \int_0^{\pi/\omega} e^{-st} \sin \omega t dt$$

$$= \frac{E}{1-e^{-\pi s/\omega}} \left[\frac{e^{-st}}{(-s)^2 + \omega^2} (-s \sin \omega t - \omega \cos \omega t) \right]_0^{\pi/\omega}$$

$$= \frac{E}{1-e^{-\pi s/\omega}} \left[\frac{e^{-s\pi/\omega}}{s^2 + \omega^2} (-s \sin \omega(\frac{\pi}{\omega}) - \omega \cos \omega(\frac{\pi}{\omega})) - \frac{e^0}{s^2 + \omega^2} (-s \sin 0 - \omega \cos 0) \right]$$

$$= \frac{E}{1-e^{-\pi s/\omega}} \left[\frac{e^{-s\pi/\omega}}{s^2 + \omega^2} (-s(0) - \omega(-1)) - \frac{1}{s^2 + \omega^2} (0 - \omega) \right]$$

$$= \frac{E}{1-e^{-\pi s/\omega}} \left[\frac{\omega e^{-s\pi/\omega}}{s^2 + \omega^2} + \frac{\omega}{s^2 + \omega^2} \right]$$

$$= \frac{E\omega (e^{-s\pi/\omega} + 1)}{s^2 + \omega^2 (1 - e^{-\pi s/\omega})}$$

5. Find L.T of triangular wave of period $2a$
 given by $f(t) = t$, $0 < t < a$
 $= 2a - t$, $a < t < 2a$

Dirac's Delta function (or) Unit Impulse function

Unit impulse function is considered as the limiting form

$$\begin{aligned} \text{of the function } f_{\epsilon}(t-a) &= 0, \text{ for } t < a \\ &= \frac{1}{\epsilon}, \text{ for } a \leq t \leq a+\epsilon \\ &= 0, \text{ for } t > a+\epsilon \end{aligned}$$

$$\text{and } \int_0^{\infty} f_{\epsilon}(t-a) dt = 1$$

$$\begin{aligned} \therefore L[f_{\epsilon}(t-a)] &= \int_0^{\infty} e^{-st} f_{\epsilon}(t-a) dt \\ &= \int_0^a e^{-st}(0) dt + \int_a^{a+\epsilon} e^{-st} \frac{1}{\epsilon} dt + \int_{a+\epsilon}^{\infty} e^{-st}(0) dt \\ &= \frac{1}{\epsilon} \int_a^{a+\epsilon} e^{-st} dt \\ &= \frac{1}{\epsilon} \left[\frac{e^{-st}}{-s} \right]_a^{a+\epsilon} \\ &= -\frac{1}{s\epsilon} \left[e^{-s(a+\epsilon)} - e^{-sa} \right] \\ &= -\frac{1}{s\epsilon} \left[e^{-sa-s\epsilon} - e^{-sa} \right] \\ &= -\frac{1}{s\epsilon} \left[e^{-sa} e^{-s\epsilon} - e^{-sa} \right] \\ &= -\frac{e^{-sa}}{s\epsilon} \left[e^{-s\epsilon} - 1 \right] \end{aligned}$$

$\therefore$ The limit of $f_{\epsilon}(t-a)$ as $\epsilon \rightarrow 0$ is denoted by $\delta(t-a)$ and is called the Dirac delta function

Laplace Transform of Dirac delta function:

$$\begin{aligned} L[\delta(t-a)] &= \lim_{\epsilon \rightarrow 0} L[f_{\epsilon}(t-a)] \\ &= \lim_{\epsilon \rightarrow 0} \frac{e^{-as}(1 - e^{-s\epsilon})}{s\epsilon} = \frac{0}{0} \\ &= e^{-as} \lim_{\epsilon \rightarrow 0} \frac{-(-s)e^{-s\epsilon}}{s(1)} = e^{-as} \lim_{\epsilon \rightarrow 0} e^{-s\epsilon} \\ &= e^{-as}(1) \quad [\because e^{-0} = 1] \quad [\text{using L. Hospital's rule}] \\ \boxed{L[\delta(t-a)] = e^{-as}} \end{aligned}$$

Thus the Dirac delta function is a generalized function defined as

$$\begin{aligned} \delta(t-a) &= \infty, \text{ when } t=a \\ &= 0, \quad t \neq a. \end{aligned}$$

Laplace Transform of Some standard functions.

1. $L[k] = \frac{k}{s}, (s > 0) \quad k \text{ is a constant.}$

$$L[k] = \int_0^{\infty} e^{-st} k \, dt$$

$$= k \left[\frac{e^{-st}}{-s} \right]_0^{\infty}$$

$$= -\frac{k}{s} [e^{-\infty} - e^{-0}]$$

$$= -\frac{k}{s} [0 - 1]$$

$$\left[\because e^{-\infty} = \frac{1}{e^{\infty}} = \frac{1}{\infty} = 0 \right]$$

$$\boxed{L[k] = \frac{k}{s}}$$

Note: i) For $k > 0, L[0] = 0$ ii) For $k=1, L[1] = \frac{1}{s}.$

2. $L[e^{at}] = \frac{1}{s-a}, \quad s-a > 0.$

$$L[e^{at}] = \int_0^{\infty} e^{-st} e^{at} \, dt$$

$$= \int_0^{\infty} e^{-(s-a)t} \, dt$$

$$= \left[\frac{e^{-(s-a)t}}{-(s-a)} \right]_0^{\infty}$$

$$= -\frac{1}{(s-a)} [e^{-\infty} - e^{-0}] = -\frac{1}{s-a} [0 - 1]$$

$$\Rightarrow \boxed{L[e^{at}] = \frac{1}{s-a}}$$

Note: $L[e^{-at}] = \frac{1}{s+a}, \quad s+a > 0.$

3. $L[\sinh at] = \frac{a}{s^2 - a^2}, \quad s > |a|, \quad a \geq 0$

$$L[\sinh at] = L\left[\frac{e^{at} - e^{-at}}{2}\right] \quad \left[\sinh x = \frac{e^x - e^{-x}}{2} \right]$$

$$= \frac{1}{2} [L(e^{at}) - L(e^{-at})] \quad [\text{By linearity property}]$$

$$= \frac{1}{2} \left[\frac{1}{s-a} - \frac{1}{s+a} \right]$$

$$= \frac{1}{2} \left[\frac{s+a - s+a}{s^2 - a^2} \right] = \frac{1}{2} \left[\frac{-2a}{s^2 - a^2} \right]$$

$$\Rightarrow \boxed{L[\sinh at] = \frac{a}{s^2 - a^2}}$$

similarly $L[\cosh at] = \frac{s}{s^2 - a^2}$, $s > |a|$, $a \geq 0$. [Hint: $\cosh x = \frac{e^x + e^{-x}}{2}$]

4. $L[\cos at] = \frac{s}{s^2 + a^2}$, $L[\sin at] = \frac{a}{s^2 + a^2}$, if $s > 0$.

w.k.t $L[e^{at}] = \frac{1}{s-a}$, Replace a by ia

$$L[e^{iat}] = \frac{1}{s-ia} = \frac{s+ia}{(s-ia)(s+ia)}$$

i.e. $L[\cos at + i \sin at] = \frac{s+ia}{s^2 + a^2}$ [$\because i^2 = -1$
 $e^{i\theta} = \cos \theta + i \sin \theta$]

$$\Rightarrow L[\cos at] + i L[\sin at] = \frac{s}{s^2 + a^2} + i \frac{a}{s^2 + a^2}$$

comparing the real and imaginary parts on both sides, we have

$$L[\cos at] = \frac{s}{s^2 + a^2} \text{ and } L[\sin at] = \frac{a}{s^2 + a^2}$$

5. $L[t^n] = \frac{\Gamma(n+1)}{s^{n+1}}$, $n+1 > 0$

$$L[t^n] = \int_0^\infty e^{-st} t^n dt$$

put $st = x$

$$= \int_0^\infty e^{-x} \left(\frac{x}{s}\right)^n \frac{dx}{s}$$

$$\Rightarrow t = \frac{x}{s}$$

$$\Rightarrow dt = \frac{1}{s} dx$$

$$= \frac{1}{s^{n+1}} \int_0^\infty e^{-x} x^n dx$$

Limits: when $t=0$, $x=0$

As $t \rightarrow \infty$, $x \rightarrow \infty$

$$= \frac{1}{s^{n+1}} \int_0^\infty e^{-x} x^{n+1-1} dx \quad \left[\because \Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx \right]$$

$$= \frac{1}{s^{n+1}} \Gamma(n+1)$$

$$\Rightarrow \boxed{L[t^n] = \frac{\Gamma(n+1)}{s^{n+1}}}$$

Note: 1) $L[t^n] = \frac{n!}{s^{n+1}}$ [$\because \Gamma(n+1) = n!$, $n=0, 1, 2, 3, \dots$]

2) $L[t^n] = \frac{n \Gamma(n)}{s^{n+1}}$ [$\Gamma(n+1) = n \Gamma(n)$, if $n > 0$]

First shifting theorem (or) First Translation.

st: If $L[f(t)] = \bar{f}(s)$, then $L[e^{at} f(t)] = \bar{f}(s-a)$, $s-a > 0$.

Proof: consider $\bar{f}(s-a) = \int_0^{\infty} e^{-(s-a)t} f(t) dt$ $\left[\bar{f}(s) = \int_0^{\infty} e^{-st} f(t) dt \right]$

$$= \int_0^{\infty} e^{-st} e^{at} f(t) dt$$

$$= \int_0^{\infty} e^{-st} \{e^{at} f(t)\} dt$$

$$= L[e^{at} f(t)]$$

Note: 1) The Laplace transform of $f(t)$ multiplied by e^{at} is obtained by replacing s by $s-a$ in $\bar{f}(s)$

2) As an application of this theorem, we obtain the following results:

i) $L[e^{at} f(t)] = \bar{f}(s+a)$

ii) $L[e^{at} \sin bt] = \frac{b}{(s-a)^2 + b^2}$

iii) $L[e^{at} \cos bt] = \frac{s-a}{(s-a)^2 + b^2}$

iv) $L[e^{at} \sinh bt] = \frac{b}{(s-a)^2 - b^2}$

v) $L[e^{at} \cosh bt] = \frac{s-a}{(s-a)^2 - b^2}$

vi) $L[e^{at} t^n] = \left[\frac{n!}{s^{n+1}} \right]_{s \rightarrow s-a} = \frac{n!}{(s-a)^{n+1}} \quad [n(n+1) = n!]$

change of scale property: st: If $L[f(t)] = \bar{f}(s)$, then $L[f(at)] = \frac{1}{a} \bar{f}\left(\frac{s}{a}\right)$

Proof: $L[f(at)] = \int_0^{\infty} e^{-st} f(at) dt$

$$= \int_0^{\infty} e^{-s\left(\frac{x}{a}\right)} f(x) \frac{dx}{a}$$

$$= \frac{1}{a} \int_0^{\infty} e^{-\left(\frac{s}{a}\right)x} f(x) dx$$

$$= \frac{1}{a} \int_0^{\infty} e^{-\frac{s}{a}t} f(t) dt$$

$$\boxed{L[f(at)] = \frac{1}{a} \bar{f}\left(\frac{s}{a}\right)}$$

put $at = x$

$$\Rightarrow t = \frac{x}{a}$$

$$\Rightarrow dt = \frac{dx}{a}$$

Limit: when $t=0$, $x=0$

As $t \rightarrow \infty$, $x \rightarrow \infty$

$$\left[\int_0^{\infty} f(x) dx = \int_0^{\infty} f(t) dt \right]$$

Multiplication by t (or) Laplace Transform of $t^n f(t)$

st: If $L[f(t)] = F(s)$ then $L[tf(t)] = -\frac{d}{ds}[F(s)]$ and

$$L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} [F(s)] \quad \text{for } n=1, 2, 3, \dots$$

proof:-

$$\text{w.k.t. } F(s) = \int_0^{\infty} e^{-st} f(t) dt$$

$$\frac{d}{ds}[F(s)] = \frac{d}{ds} \left[\int_0^{\infty} e^{-st} f(t) dt \right]$$

$$= \int_0^{\infty} -t e^{-st} f(t) dt$$

$$= - \int_0^{\infty} e^{-st} [t f(t)] dt$$

$$= -L[tf(t)]$$

$$\Rightarrow \frac{d}{ds}[F(s)] = -L[tf(t)]$$

$$\therefore L[tf(t)] = -\frac{d}{ds}[F(s)] \quad \text{--- (1)}$$

$$L[t^2 f(t)] = L[t \cdot tf(t)]$$

$$= -\frac{d}{ds} [L[tf(t)]]$$

$$= -\frac{d}{ds} \left[-\frac{d}{ds} [F(s)] \right] \quad [\because (1)]$$

$$\text{i.e. } L[t^2 f(t)] = (-1)^2 \frac{d^2}{ds^2} [F(s)]$$

$$\text{H}y \quad L[t^3 f(t)] = (-1)^3 \frac{d^3}{ds^3} [F(s)]$$

$$\text{In general } \boxed{L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} [F(s)]}$$

Division by t

st: If $L[f(t)] = F(s)$, then $L\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s) ds$, provided the integral exists.

proof: w.k.t. $F(s) = \int_0^\infty e^{-st} f(t) dt$

Integrating on both sides w.r.t. s from s to ∞ , we have

$$\begin{aligned} \int_s^\infty F(s) ds &= \int_s^\infty \int_0^\infty e^{-st} f(t) dt ds \\ &= \int_0^\infty f(t) \left[\int_s^\infty e^{-st} ds \right] dt \\ &= \int_0^\infty f(t) \left[\frac{e^{-st}}{-t} \right]_s^\infty dt \\ &= \int_0^\infty f(t) \left[\frac{e^{-\infty} - e^{-st}}{-t} \right] dt \\ &= \int_0^\infty f(t) \left[0 + \frac{e^{-st}}{t} \right] dt \\ &= \int_0^\infty e^{-st} \frac{f(t)}{t} dt \end{aligned}$$

$$\int_s^\infty F(s) ds = L\left[\frac{f(t)}{t}\right]$$

$$\therefore L\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s) ds$$

$$\begin{aligned} \text{Now } L\left[\frac{f(t)}{t^2}\right] &= L\left[\frac{f(t)}{t}\right] = \int_s^\infty L\left[\frac{f(t)}{t}\right] ds \\ &= \int_s^\infty \left[\int_s^\infty L[f(t)] ds \right] ds \\ &= \int_s^\infty \int_s^\infty F(s) ds ds \end{aligned}$$

$$\text{If } L\left[\frac{f(t)}{t^3}\right] = \int_s^\infty \int_s^\infty \int_s^\infty F(s) ds ds ds$$

$$\text{In general } L\left[\frac{f(t)}{t^n}\right] = \int_s^\infty \int_s^\infty \dots \int_s^\infty F(s) ds ds \dots ds \quad (n \text{ times})$$

Laplace Trans-form of derivatives

St: If $f(t)$ is continuous and of exponential order and $f'(t)$ is piecewise continuous then,

$$L[f'(t)] = s \bar{f}(s) - f(0) \quad \text{where } \bar{f}(s) = L[f(t)].$$

$$\text{and } L[f^n(t)] = s^n \bar{f}(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \dots - f^{(n-1)}(0).$$

proof - Given $f(t)$ is an exponential order $\lim_{t \rightarrow \infty} e^{-st} f(t) = 0$ - (1)

and $f'(t)$ is piecewise continuous $\forall t$.

By the def. $L[f'(t)] = \int_0^{\infty} e^{-st} f'(t) dt$, Integration by parts

$$= \left[e^{-st} f(t) \right]_0^{\infty} - \int_0^{\infty} e^{-st} (-s) f(t) dt$$

$$= [0 - e^0 f(0)] + s \int_0^{\infty} e^{-st} f(t) dt$$

$$= -f(0) + s \bar{f}(s)$$

$$\Rightarrow L[f'(t)] = s \bar{f}(s) - f(0)$$

$$\text{If } L[f''(t)] = L[g'(t)] \quad \text{where } f'(t) = g(t)$$

$$= s L[g(t)] - g(0)$$

$$= s L[f'(t)] - f'(0)$$

$$= s [s \bar{f}(s) - f(0)] - f'(0)$$

$$L[f''(t)] = s^2 \bar{f}(s) - s f(0) - f'(0)$$

$$\text{and } L[f'''(t)] = s^3 \bar{f}(s) - s^2 f(0) - s f'(0) - f''(0)$$

$$\text{In general } L[f^n(t)] = s^n \bar{f}(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \dots - f^{(n-1)}(0)$$

Note: 1) $L[f^n(t)] = s^n \bar{f}(s)$, if $f(0) = f'(0) = \dots = f^{(n-1)}(0) = 0$.

$$2) L[f'''(t)] = L[(f''(t))'] = s L[f''(t)] - f''(0)$$

$$= s [s^2 \bar{f}(s) - s f(0) - f'(0)] - f''(0)$$

$$= s^3 \bar{f}(s) - s^2 f(0) - s f'(0) - f''(0)$$

Laplace Trans-form of Integrals

Th: If $L[f(t)] = \bar{F}(s)$ then $L\left[\int_0^t f(u) du\right] = \frac{\bar{F}(s)}{s}$

proof: Let $g(t) = \int_0^t f(u) du$, $g'(t) = \frac{d}{dt}\left[\int_0^t f(u) du\right] = f(t)$

$$\text{and } g(0) = \int_0^0 f(u) du = 0$$

$$\text{Now } L[g'(t)] = s L[g(t)] - g(0)$$

$$\text{ie. } L[f(t)] = s L[g(t)] - 0$$

$$\therefore L[g(t)] = \frac{L[f(t)]}{s}$$

$$\text{i.e. } L\left[\int_0^t f(u) du\right] = \frac{\bar{F}(s)}{s}$$

$$\text{similarly } L\left[\int_0^t \int_0^t f(u) du du\right] = \frac{\bar{F}(s)}{s^2}$$

$$\text{In general, } L\left[\int_0^t \int_0^t \dots \int_0^t f(u) du du \dots du\right] = \frac{\bar{F}(s)}{s^n}$$

(n times) (n times)

Second Translation (or) Second shifting theorem.

Th: If $L[f(t)] = \bar{F}(s)$ and $g(t) = f(t-a)$, $t > a$ then
 $= 0$, $t < a$

$$L[g(t)] = e^{-as} \bar{F}(s).$$

proof: By def.

$$\begin{aligned} L[g(t)] &= \int_0^\infty e^{-st} g(t) dt \\ &= \int_0^a e^{-st} \cdot 0 dt + \int_a^\infty e^{-st} f(t-a) dt \\ &= \int_a^\infty e^{-st} f(t-a) dt \\ &= \int_0^\infty e^{-s(u+a)} f(u) du \quad \begin{array}{l} \text{put } t-a = u \\ \Rightarrow t = u+a \\ \Rightarrow dt = du \end{array} \\ &= \int_0^\infty e^{-su} e^{-sa} f(u) du \quad \begin{array}{l} \text{Limits: when } t=a, u=0 \\ \text{As } t \rightarrow \infty, u \rightarrow \infty \end{array} \\ &= e^{-as} \int_0^\infty e^{-su} f(u) du \\ \Rightarrow L[g(t)] &= e^{-as} \int_0^\infty e^{-st} f(t) dt \quad \left[\because \int_0^a f(x) dx = \int_0^a f(t) dt \right] \\ \therefore L[g(t)] &= e^{-as} \bar{F}(s) \end{aligned}$$

Second Translation (or) Second shifting theorem

Th: If $L[f(t)] = \bar{F}(s)$ and $g(t) = f(t-a)u(t-a) = 0 \text{ if } t < a$
 $= f(t-a) \text{ if } t > a$

then $L[g(t)] = L[f(t-a)u(t-a)] = e^{-as} \bar{F}(s)$

Proof:-

$$\begin{aligned}
 L[g(t)] &= \int_0^{\infty} e^{-st} g(t) dt \\
 &= \int_0^a e^{-st} g(t) dt + \int_a^{\infty} e^{-st} g(t) dt \\
 &= 0 + \int_a^{\infty} e^{-st} f(t-a) dt \\
 &= \int_a^{\infty} e^{-st} f(t-a) dt \quad \text{put } t-a = u \\
 &= \int_0^{\infty} e^{-s(u+a)} f(u) du \quad \Rightarrow t = u+a \\
 &= \int_0^{\infty} e^{-sa} e^{-su} f(u) du \quad \Rightarrow dt = du \\
 &= e^{-as} \int_0^{\infty} e^{-su} f(u) du \quad \text{Limits: when } t=a, u=0 \\
 &= e^{-as} \int_0^{\infty} e^{-st} f(t) dt \quad \text{As } t \rightarrow \infty, u \rightarrow \infty \\
 &= e^{-as} \int_0^{\infty} e^{-st} f(t) dt \quad \left[\because \int_0^a f(x) dx = \int_0^a f(t) dt \right]
 \end{aligned}$$

$L[g(t)] = e^{-as} \bar{F}(s)$

Inverse Laplace Transforms

Inverse L.T :- If $L[f(t)] = \bar{f}(s)$ then $\bar{f}(t) = L^{-1}[\bar{f}(s)]$ is called the inverse Laplace Transform of $\bar{f}(s)$.

Here the symbol L^{-1} which transforms $\bar{f}(s)$ to $f(t)$ can be called Inverse Laplace Transform operator.

* Inverse L.T of some standard functions:-

Laplace Transform

Inverse Laplace Transform

$$1. L[K] = \frac{K}{s}$$

$$L^{-1}\left[\frac{K}{s}\right] = K$$

$$2. L[t^n] = \frac{n!}{s^{n+1}}$$

$$L^{-1}\left[\frac{1}{s^{n+1}}\right] = \frac{t^n}{n!}$$

$$L[t^n] = \frac{n!}{s^{n+1}}$$

$$L^{-1}\left[\frac{1}{s^{n+1}}\right] = \frac{t^n}{n!}$$

$$3. L[e^{at}] = \frac{1}{s-a}$$

$$L^{-1}\left[\frac{1}{s-a}\right] = e^{at}$$

$$4. L[e^{at}] = \frac{1}{s-a}$$

$$L^{-1}\left[\frac{1}{s-a}\right] = e^{at}$$

$$5. L[\sinh at] = \frac{a}{s^2 - a^2}$$

$$L^{-1}\left[\frac{a}{s^2 - a^2}\right] = \sinh at$$

$$6. L[\cosh at] = \frac{s}{s^2 - a^2}$$

$$L^{-1}\left[\frac{s}{s^2 - a^2}\right] = \cosh at$$

$$7. L[\sin at] = \frac{a}{s^2 + a^2}$$

$$L^{-1}\left[\frac{a}{s^2 + a^2}\right] = \sin at$$

$$8. L[\cos at] = \frac{s}{s^2 + a^2}$$

$$L^{-1}\left[\frac{s}{s^2 + a^2}\right] = \cos at$$

∴ The inverse Laplace Transform of a function $\bar{f}(s)$ can be obtained either by using the above standard results or by splitting the given function into partial fractions and then

using above standard results.

1. Find $\mathcal{L}^{-1} \left[\frac{s^2 - 3s + 4}{s^3} \right]$

$$\begin{aligned} \text{Given } \mathcal{L}^{-1} \left[\frac{s^2 - 3s + 4}{s^3} \right] &= \mathcal{L}^{-1} \left[\frac{s^2}{s^3} - \frac{3s}{s^3} + \frac{4}{s^3} \right] \\ &= \mathcal{L}^{-1} \left[\frac{1}{s} - \frac{3}{s^2} + \frac{4}{s^3} \right] = \mathcal{L}^{-1} \left(\frac{1}{s} \right) - \mathcal{L}^{-1} \left(\frac{3}{s^2} \right) + 4 \mathcal{L}^{-1} \left(\frac{1}{s^3} \right) \\ &= 1 - 3 \mathcal{L}^{-1} \left(\frac{1}{s^2} \right) + 4 \mathcal{L}^{-1} \left(\frac{1}{s^3} \right) \\ &= 1 - 3 \frac{t^1}{1!} + 4 \frac{t^2}{2!} = 1 - 3t + 2t^2 = f(t) \end{aligned}$$

2. Find $\mathcal{L}^{-1} \left[\frac{2s-5}{4s^2+25} + \frac{4s-18}{9-s^2} \right]$

$$\begin{aligned} \text{Given } \mathcal{L}^{-1} \left[\frac{2s-5}{4s^2+25} + \frac{4s-18}{9-s^2} \right] &= \mathcal{L}^{-1} \left[\frac{2s}{4s^2+25} - \frac{5}{4s^2+25} + \frac{4s}{9-s^2} - \frac{18}{9-s^2} \right] \\ &= \mathcal{L}^{-1} \left[\frac{2s}{4(s^2 + \frac{25}{4})} - \frac{5}{4(s^2 + \frac{25}{4})} + \frac{4s}{3^2 - s^2} - \frac{18}{3^2 - s^2} \right] \\ &= \frac{1}{2} \mathcal{L}^{-1} \left[\frac{s}{s^2 + (\frac{5}{2})^2} \right] - \frac{5}{4} \mathcal{L}^{-1} \left[\frac{1}{s^2 + (\frac{5}{2})^2} \right] - 4 \mathcal{L}^{-1} \left(\frac{s}{s^2 - 3^2} \right) + 18 \mathcal{L}^{-1} \left(\frac{1}{s^2 - 3^2} \right) \\ &= \frac{1}{2} \cos \frac{5}{2} t - \frac{1}{2} \mathcal{L}^{-1} \left[\frac{5/2}{s^2 + (\frac{5}{2})^2} \right] - 4 \cosh 3t + 6 \mathcal{L}^{-1} \left(\frac{3}{s^2 - 3^2} \right) \\ &= \frac{1}{2} \cos \frac{5}{2} t - \frac{1}{2} \sin \frac{5}{2} t - 4 \cosh 3t + 6 \sinh 3t. \end{aligned}$$

$$3) \quad \mathcal{L}^{-1} \left[\frac{3(s^2-2)^2}{2s^5} \right]$$

$$= \frac{3}{2} \mathcal{L}^{-1} \left[\frac{s^4 - 4s^2 + 4}{s^5} \right]$$

$$= \frac{3}{2} \mathcal{L}^{-1} \left[\frac{1}{s} - \frac{4}{s^3} + \frac{4}{s^5} \right]$$

$$= \frac{3}{2} \left[\mathcal{L}^{-1} \left(\frac{1}{s} \right) - 4 \mathcal{L}^{-1} \left(\frac{1}{s^3} \right) + 4 \mathcal{L}^{-1} \left(\frac{1}{s^5} \right) \right]$$

$$= \frac{3}{2} \left[1 - 4 \frac{t^2}{2!} + 4 \frac{t^4}{4!} \right] \quad \left[\mathcal{L}^{-1} \left[\frac{1}{s^{n+1}} \right] = \frac{t^n}{n!} \right]$$

$$= \frac{3}{2} \left[1 - 2t^2 + \frac{t^4}{6} \right] = f(t)$$

$$4) \quad \mathcal{L}^{-1} \left[\left(\frac{\sqrt{s}-1}{s} \right)^2 \right] = \mathcal{L}^{-1} \left[\frac{s+1-2\sqrt{s}}{s^2} \right]$$

$$= \mathcal{L}^{-1} \left[\frac{1}{s} + \frac{1}{s^2} - 2 \frac{s^{\frac{1}{2}}}{s^2} \right]$$

$$= \mathcal{L}^{-1} \left[\frac{1}{s} \right] + \mathcal{L}^{-1} \left[\frac{1}{s^2} \right] - 2 \mathcal{L}^{-1} \left[\frac{1}{s^{\frac{3}{2}}} \right]$$

$$= 1 + \frac{t}{1!} - 2 \mathcal{L}^{-1} \left[\frac{1}{s^{\frac{3}{2}}} \right]$$

$$= 1 + t - 2 \mathcal{L}^{-1} \left[\frac{1}{s^{\frac{1}{2}+1}} \right] \quad \left[\mathcal{L}^{-1} \left[\frac{1}{s^{n+1}} \right] = \frac{t^n}{\Gamma(n+1)} \right]$$

$$= 1 + t - 2 \frac{t^{\frac{1}{2}}}{\Gamma(\frac{1}{2}+1)}$$

$$= 1 + t - 2 \frac{\sqrt{t}}{\frac{1}{2}\Gamma(\frac{1}{2})} = 1 + t - 4 \frac{\sqrt{t}}{\sqrt{\pi}} = f(t)$$

$$5. \quad \mathcal{L}^{-1}\left[\frac{1}{s} e^{-\frac{1}{s}}\right]$$

$$\bar{f}(s) = \frac{1}{s} e^{-\frac{1}{s}} \left[e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots \right]$$

$$= \frac{1}{s} \left[1 + \frac{\left(-\frac{1}{s}\right)}{1!} + \frac{\left(-\frac{1}{s}\right)^2}{2!} + \dots \right]$$

$$= \frac{1}{s} \left[1 - \frac{1}{s} + \frac{1}{s^2 \cdot 2} - \dots \right]$$

$$= \frac{1}{s} - \frac{1}{s^2} + \frac{1}{2} \frac{1}{s^3} - \dots$$

$$\therefore \mathcal{L}^{-1}[\bar{f}(s)] = \mathcal{L}^{-1}\left[\frac{1}{s}\right] - \mathcal{L}^{-1}\left[\frac{1}{s^2}\right] + \frac{1}{2} \mathcal{L}^{-1}\left[\frac{1}{s^3}\right] - \dots$$

$$= 1 - \frac{t}{1!} + \frac{1}{2} \frac{t^2}{2!} - \dots = f(t)$$

$$6. \quad \mathcal{L}^{-1}\left[\frac{1}{s} \sin \frac{1}{s}\right]$$

$$\bar{f}(s) = \frac{1}{s} \sin\left(\frac{1}{s}\right) \left[\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \right]$$

$$= \frac{1}{s} \left[\frac{1}{s} - \frac{\left(\frac{1}{s}\right)^3}{3!} + \frac{\left(\frac{1}{s}\right)^5}{5!} - \dots \right]$$

$$= \frac{1}{s} \left[\frac{1}{s} - \frac{1}{3! s^3} + \frac{1}{5! s^5} - \dots \right]$$

$$= \frac{1}{s^2} - \frac{1}{3!} \frac{1}{s^4} + \frac{1}{5!} \frac{1}{s^6} - \dots$$

$$\therefore \mathcal{L}^{-1}[\bar{f}(s)] = \mathcal{L}^{-1}\left[\frac{1}{s^2}\right] - \frac{1}{3!} \mathcal{L}^{-1}\left[\frac{1}{s^4}\right] + \frac{1}{5!} \mathcal{L}^{-1}\left[\frac{1}{s^6}\right] - \dots$$

$$= \frac{t^1}{1!} - \frac{1}{3!} \frac{t^3}{3!} + \frac{1}{5!} \frac{t^5}{5!} - \dots$$

$$= f(t)$$

4) Problems based on partial fractions:

1) Find $L^{-1} \left(\frac{1}{s^2 - 5s + 6} \right)$

Sol $L^{-1} \left(\frac{1}{(s-2)(s-3)} \right)$

$$F(s) = s^2 - 5s + 6$$

$$= s^2 - 2s - 3s + 6$$

$$= s(s-2) - 3(s-2)$$

$$= (s-2)(s-3)$$

let $\bar{f}(s) = \frac{1}{(s-2)(s-3)} = \frac{A}{s-2} + \frac{B}{s-3}$

$$1 = A(s-3) + B(s-2)$$

Put $s = 3$

$$1 = B \Rightarrow B = 1$$

Put $s = 2$

$$1 = A(-1) \Rightarrow A = -1$$

$$\therefore \bar{f}(s) = \frac{1}{s-3} - \frac{1}{s-2}$$

$$L^{-1} \left(\frac{1}{s-3} - \frac{1}{s-2} \right) = L^{-1} \left(\frac{1}{s-3} \right) - L^{-1} \left(\frac{1}{s-2} \right)$$

$$= e^{3t} - e^{2t}$$

2) Find $L^{-1} \left(\frac{2s^2 - 6s + 5}{s^3 - 6s^2 + 11s - 6} \right)$

Sol let $\bar{f}(s) = \frac{2s^2 - 6s + 5}{s^3 - 6s^2 + 11s - 6}$

Take denominator for factors

$$s^3 - 6s^2 + 11s - 6$$

Put $s = 1$ by trial and error method

$$1 - 6 + 11 - 6 \Rightarrow 12 - 12 = 0$$

By synthetic division $s=1$

$$\begin{array}{r|rrrr} 1 & 1 & -6 & 11 & -6 \\ & & 0 & 1 & -5 & 6 \\ \hline & 1 & -5 & 6 & 10 \end{array}$$

$$(s-1)(s^2-5s+6)$$

$$(s-1)(s-2)(s-3)$$

$$\therefore \bar{f}(s) = \frac{2s^2-6s+5}{(s-1)(s-2)(s-3)} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s-3}$$

$$\therefore 2s^2-6s+5 = A(s-2)(s-3) + B(s-1)(s-3) + C(s-1)(s-2)$$

Put $s=1$ then

$$2-6+5 = A(1)(2)$$

$$1 = 2A \Rightarrow A = \frac{1}{2}$$

Put $s=2$ then

$$8-12+5 = B(1)(-1)$$

$$-1 = B$$

Put $s=3$

$$18-18+5 = C(2)(1)$$

$$C = \frac{5}{2}$$

$$\therefore \bar{f}(s) = \frac{1/2}{s-1} - \frac{1}{s-2} + \frac{5/2}{s-3}$$

$$\therefore \mathcal{L}^{-1}\left(\frac{2s^2-6s+5}{(s-1)(s-2)(s-3)}\right) = \mathcal{L}^{-1}\left(\frac{1/2}{s-1}\right) - \mathcal{L}^{-1}\left(\frac{1}{s-2}\right) + \frac{5}{2}\mathcal{L}^{-1}\left(\frac{1}{s-3}\right)$$

$$= \frac{1}{2}e^t - e^{2t} + \frac{5}{2}e^{3t}$$

43. Find $\mathcal{L}^{-1} \left[\frac{s^2 + 2s - 4}{(s-5)(s^2+9)} \right]$

Given $\mathcal{L}^{-1} \left[\frac{s^2 + 2s - 4}{(s-5)(s^2+9)} \right]$ let $\bar{f}(s) = \frac{s^2 + 2s - 4}{(s-5)(s^2+9)} = \frac{A}{s-5} + \frac{Bs+C}{s^2+9}$

$$\Rightarrow s^2 + 2s - 4 = A(s^2 + 9) + (Bs + C)(s - 5) \quad \text{--- (1)}$$

put $s = 5$ in (1) we get

$$31 = A(34) + 0 \quad \Rightarrow A = 31/34$$

Comparing Coefficients of s^2, s on both sides we get

$$1 = A + B \quad \Rightarrow B = 1 - A = 1 - \frac{31}{34} = \frac{3}{34}$$

$$2 = -5B + C \quad \Rightarrow C = 2 + 5\left(\frac{3}{34}\right) = \frac{68+15}{34} = \frac{83}{34}$$

$$\therefore \frac{s^2 + 2s - 4}{(s-5)(s^2+9)} = \frac{31/34}{s-5} + \frac{\frac{3}{34}s + \frac{83}{34}}{s^2+9}$$

$$\therefore \mathcal{L}^{-1} \left[\frac{s^2 + 2s - 4}{(s-5)(s^2+9)} \right] = \mathcal{L}^{-1} \left[\frac{31/34}{s-5} + \frac{\frac{3}{34}s + \frac{83}{34}}{s^2+9} \right]$$

$$= \frac{31}{34} \mathcal{L}^{-1} \left[\frac{1}{s-5} \right] + \frac{3}{34} \mathcal{L}^{-1} \left[\frac{s}{s^2+9} \right] + \frac{83}{34} \mathcal{L}^{-1} \left[\frac{1}{s^2+9} \right]$$

$$= \frac{31}{34} e^{5t} + \frac{3}{34} \mathcal{L}^{-1} \left[\frac{s}{s^2+3^2} \right] + \frac{83}{34 \times 3} \mathcal{L}^{-1} \left[\frac{3}{s^2+3^2} \right]$$

$$= \frac{31}{34} e^{5t} + \frac{3}{34} \cos 3t + \frac{83}{102} \sin 3t$$

Q) Find $L^{-1} \left[\frac{s^2}{(s+2)^3} \right]$

Sol) $\bar{f}(s) = \frac{s^2}{(s+2)^3} = \frac{A}{s+2} + \frac{B}{(s+2)^2} + \frac{C}{(s+2)^3}$

$$s^2 = A(s+2)^2 + B(s+2) + C$$

Put $s = -2$ in above eq we get

$$\boxed{4 = C}$$

Comparing the co-efficients of s^2 on both sides

$$\boxed{1 = A}$$

Comparing the co-efficients of s on both sides

$$4A + B = 0$$

$$\boxed{B = -4}$$

$$\frac{s^2}{(s+2)^3} = \frac{1}{s+2} - \frac{4}{(s+2)^2} + \frac{4}{(s+2)^3}$$

$$\therefore \mathcal{L}^{-1}\left[\frac{s^2}{(s+2)^3}\right] = \mathcal{L}^{-1}\left[\frac{1}{s+2}\right] - 4\mathcal{L}^{-1}\left[\frac{1}{(s+2)^2}\right] + 4\mathcal{L}^{-1}\left[\frac{1}{(s+2)^3}\right]$$

$$= e^{-2t} - 4\mathcal{L}^{-1}[\tilde{f}(s+2)] + 4\mathcal{L}^{-1}[\tilde{f}(s+2)]$$

$\mathcal{L}^{-1}[\tilde{f}(s+a)] = e^{-at} \mathcal{L}^{-1}[\tilde{f}(s)]$

$$= e^{-2t} - 4 \cdot e^{-2t} \mathcal{L}^{-1}\left[\frac{1}{s^2}\right] + 4e^{-2t} \mathcal{L}^{-1}\left[\frac{1}{s^3}\right]$$

$$= e^{-2t} - 4e^{-2t} \cdot t + 4e^{-2t} \cdot \frac{t^2}{2!}$$

$$= e^{-2t} [2t^2 - 4t + 1]$$

$$6) \mathcal{L}^{-1} \left[\frac{s}{s^4 + s^2 + 1} \right]$$

Hint:

$$\begin{aligned} \bar{f}(s) &= \frac{s}{s^4 + s^2 + 1} \\ &= \frac{s}{(s^2 + s + 1)(s^2 - s + 1)} \end{aligned} \quad \left\{ \begin{aligned} s^4 + s^2 + 1 &= s^4 + 2s^2 - s^2 + 1 \\ &= (s^2)^2 + 2s^2(1) + 1^2 - s^2 \\ &= (s^2 + 1)^2 - s^2 \\ &= (s^2 + 1 + s)(s^2 + 1 - s) \end{aligned} \right.$$

$$= \frac{1}{2} \left[\frac{s^2 + s + 1 - (s^2 - s + 1)}{(s^2 + s + 1)(s^2 - s + 1)} \right]$$

$$\bar{f}(s) = \frac{1}{2} \left[\frac{1}{s^2 - s + 1} - \frac{1}{s^2 + s + 1} \right]$$

$$\mathcal{L}^{-1}[\bar{f}(s)] = \frac{1}{2} \mathcal{L}^{-1} \left[\frac{1}{s^2 - s + 1} \right] - \frac{1}{2} \mathcal{L}^{-1} \left[\frac{1}{s^2 + s + 1} \right]$$

$$= \frac{1}{2} \mathcal{L}^{-1} \left[\frac{1}{s^2 - 2s(\frac{1}{2}) + (\frac{1}{2})^2 - (\frac{1}{2})^2 + 1} \right] \quad \left[-\frac{1}{4} + 1 \right]$$

$$- \frac{1}{2} \mathcal{L}^{-1} \left[\frac{1}{s^2 + 2s(\frac{1}{2}) + (\frac{1}{2})^2 - (\frac{1}{2})^2 + 1} \right]$$

$$= \frac{1}{2} \mathcal{L}^{-1} \left[\frac{1}{(s - \frac{1}{2})^2 + \frac{3}{4}} \right] - \frac{1}{2} \mathcal{L}^{-1} \left[\frac{1}{(s + \frac{1}{2})^2 + \frac{3}{4}} \right]$$

$$= \frac{1}{2} \mathcal{L}^{-1} \left[\frac{1}{(s - \frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2} \right] - \frac{1}{2} \mathcal{L}^{-1} \left[\frac{1}{(s + \frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2} \right]$$

$$= \frac{1}{2} \mathcal{L}^{-1} [\bar{f}(s - \frac{1}{2})] - \frac{1}{2} \mathcal{L}^{-1} [\bar{f}(s + \frac{1}{2})]$$

$$= \frac{1}{2} e^{\frac{1}{2}t} \mathcal{L}^{-1} [\bar{f}(s)] - \frac{1}{2} \mathcal{L}^{-1} [\bar{f}(s)] e^{-\frac{1}{2}t}$$

$$= \frac{1}{2} e^{\frac{1}{2}t} \mathcal{L}^{-1} \left[\frac{1}{s^2 + (\frac{\sqrt{3}}{2})^2} \right] - \frac{1}{2} e^{-\frac{1}{2}t} \mathcal{L}^{-1} \left[\frac{1}{s^2 + (\frac{\sqrt{3}}{2})^2} \right]$$

$$= \frac{1}{2} e^{\frac{1}{2}t} \frac{\sin(\frac{\sqrt{3}}{2}t)}{\frac{\sqrt{3}}{2}} - \frac{1}{2} e^{-\frac{1}{2}t} \frac{\sin(\frac{\sqrt{3}}{2}t)}{\frac{\sqrt{3}}{2}}$$

$$= \frac{1}{\sqrt{3}} \sin \frac{\sqrt{3}}{2} t \left[e^{\frac{1}{2}t} - e^{-\frac{1}{2}t} \right]$$

$$7) \mathcal{L}^{-1} \left[\frac{s}{s^4 + 4a^4} \right]$$

$$\bar{f}(s) = \frac{s}{s^4 + a^4}, \quad s^4 + 4a^4 = (s^2)^2 + (2a^2)^2$$

$$= (s^2 + 2a^2)^2 - 2s^2 a^2 (2)$$

$$= (s^2 + 2a^2)^2 - (2as)^2$$

$$= (s^2 + 2a^2 + 2as)(s^2 + 2a^2 - 2as)$$

$$\bar{f}(s) = \frac{s}{(s^2 + 2as + 2a^2)(s^2 - 2as + 2a^2)}$$

$$= \frac{1}{4a} \left[\frac{1}{s^2 + 2} - \frac{1}{s^2 - 2as + 2a^2} - \frac{1}{s^2 + 2as + 2a^2} \right]$$

$$8) \mathcal{L}^{-1} \left[\frac{1}{s^3 - 1} \right] \quad \text{Hint}$$

$$s^3 - 1^3 = (s-1)(s^2 + s + 1)$$

$$\bar{f}(s) = \frac{1}{(s-1)(s^2 + s + 1)} = \frac{A}{s-1} + \frac{Bs+C}{s^2 + s + 1}$$

$$Q4. \mathcal{L}^{-1} \left[\frac{s^2}{(s^2+4)(s^2+9)} \right]$$

$$F(s) = \frac{s^2}{(s^2+4)(s^2+9)}$$

$$= s^2 \left[\frac{1}{(s^2+4)(s^2+9)} \right]$$

$$= \frac{s^2}{2} \left[\frac{1}{s^2+4} - \frac{1}{s^2+9} \right]$$

$$= \frac{1}{2} \left[\frac{s^2}{s^2+4} - \frac{s^2}{s^2+9} \right]$$

$$= \frac{1}{2} \left[\frac{s^2+4-4}{s^2+4} - \frac{(s^2+9-9)}{s^2+9} \right]$$

$$= \frac{1}{2} \left[\cancel{s^2} - \frac{4}{s^2+4} - \cancel{s^2} + \frac{9}{s^2+9} \right]$$

$$F(s) = \frac{1}{2} \left[-\frac{4}{s^2+4} + \frac{9}{s^2+9} \right]$$

$$\Rightarrow F(s) = -\frac{2}{2} \frac{2}{s^2+2^2} + \frac{3}{2} \frac{3}{s^2+3^2}$$

$$\therefore \mathcal{L}^{-1}[F(s)] = -\frac{2}{2} \mathcal{L}^{-1} \left[\frac{2}{s^2+2^2} \right] + \frac{3}{2} \mathcal{L}^{-1} \left[\frac{3}{s^2+3^2} \right]$$

$$= -\frac{2}{2} \sin 2t + \frac{3}{2} \sin 3t = f(t)$$

$$\left[\mathcal{L}^{-1} \left[\frac{a}{s^2+a^2} \right] = \sin at \right]$$

Alternatively put $s^2 = p$

$$\mathcal{L}^{-1} \left[\frac{p}{(p+4)(p+9)} \right]$$

First Shifting Theorem:-

If $\mathcal{L}^{-1}[\tilde{f}(s)] = f(t)$ then (i) $\mathcal{L}^{-1}[\tilde{f}(s+a)] = e^{-at} f(t)$

(ii) $\mathcal{L}^{-1}[\tilde{f}(s-a)] = e^{at} f(t)$

1. Find $\mathcal{L}^{-1}\left[\frac{1}{(s+2)^2+16}\right]$

Soln $\mathcal{L}^{-1}\left[\frac{1}{(s+2)^2+16}\right] = \mathcal{L}^{-1}[\tilde{f}(s+2)]$ [By F.S.T. $\mathcal{L}^{-1}[\tilde{f}(s+a)] = e^{-at} \mathcal{L}^{-1}[\tilde{f}(s)]$]
 $= e^{-2t} \mathcal{L}^{-1}[\tilde{f}(s)]$
 $= e^{-2t} \mathcal{L}^{-1}\left[\frac{1}{s^2+16}\right] = e^{-2t} \frac{1}{4} \sin 4t$
 $= \frac{1}{4} e^{-2t} \sin 4t = f(t)$ [∵ $\mathcal{L}^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a} \sin at$]

2. Find $\mathcal{L}^{-1}\left[\frac{s+2}{(s-2)^3}\right]$

$\mathcal{L}^{-1}\left[\frac{s+2}{(s-2)^3}\right] = \mathcal{L}^{-1}\left[\frac{(s-2)+4}{(s-2)^3}\right] = \mathcal{L}^{-1}[\tilde{f}(s-2)]$ [By F.S.T. $\mathcal{L}^{-1}[\tilde{f}(s-a)] = e^{at} \mathcal{L}^{-1}[\tilde{f}(s)]$]
 $= e^{2t} \mathcal{L}^{-1}[\tilde{f}(s)] = e^{2t} \mathcal{L}^{-1}\left[\frac{s+4}{s^3}\right]$
 $= e^{2t} \left[\mathcal{L}^{-1}\left[\frac{s}{s^3}\right] + 4 \mathcal{L}^{-1}\left[\frac{1}{s^3}\right] \right]$
 $= e^{2t} \left[\mathcal{L}^{-1}\left[\frac{1}{s^2}\right] + 4 \mathcal{L}^{-1}\left[\frac{1}{s^3}\right] \right] = e^{2t} \left[t + 4 \frac{t^2}{2} \right]$
 $= e^{2t} (t + 2t^2) = f(t)$

3. Find $\mathcal{L}^{-1}\left[\frac{3s-2}{s^2-4s+20}\right]$

$\mathcal{L}^{-1}\left[\frac{3s-2}{s^2-4s+20}\right] = \mathcal{L}^{-1}\left[\frac{3s-2}{(s-2)^2+16}\right] = \mathcal{L}^{-1}\left[\frac{3(s-2)+4}{(s-2)^2+16}\right]$
 $= \mathcal{L}^{-1}\left[\frac{3(s-2)+4}{(s-2)^2+16}\right]$ [By F.S.T. $\mathcal{L}^{-1}[\tilde{f}(s-a)] = e^{at} \mathcal{L}^{-1}[\tilde{f}(s)]$]
 $= \mathcal{L}^{-1}[\tilde{f}(s-2)] = e^{2t} \mathcal{L}^{-1}[\tilde{f}(s)]$

4. Find $\mathcal{L}^{-1}\left[\frac{2s+12}{s^2+6s+13}\right]$

$\mathcal{L}^{-1}\left[\frac{2s+12}{s^2+6s+13}\right]$

$= e^{-3t} \left(2 \mathcal{L}^{-1}\left[\frac{s}{s^2+6s+13}\right] + 6 \mathcal{L}^{-1}\left[\frac{1}{s^2+6s+13}\right] \right)$

* Convolution

Def:- Let $f(t)$ and $g(t)$ be functions defined for $t > 0$.

$g(t)$ is defined for $t > 0$.

Properties:-

(i) Convolution

$f(t) * g(t)$

(ii) Convolution

$f(t) * g(t)$

(iii) $f(t) * c$

Second Shifting Theorem:-

If $L^{-1}[\bar{f}(s)] = f(t)$, then $L^{-1}[e^{-as} \bar{f}(s)] = g(t)$

Where $g(t) = f(t-a)$ if $t > a$

$= 0$ if $t < a$

(or)

$L^{-1}[e^{-as} \bar{f}(s)] = f(t-a)$ if $t > a$

$= 0$ if $t < a$

Sums

1) Find $L^{-1}\left[\frac{e^{-2s}}{s^2 + 8s + 25}\right]$

Sol The given problem is of the form $L^{-1}[e^{-as} \cdot \bar{f}(s)]$

$$\bar{F}(s) = \frac{1}{s^2 + 8s + 25}$$

$$\mathcal{L}^{-1}(\bar{F}(s)) = \mathcal{L}^{-1}\left(\frac{1}{s^2 + 8s + 25}\right)$$

$$= \mathcal{L}^{-1}\left(\frac{1}{s^2 + 2 \cdot s \cdot 4 + 4^2 + 9}\right) = \mathcal{L}^{-1}\left(\frac{1}{(s+4)^2 + 9}\right)$$

$$= \mathcal{L}^{-1}[\bar{f}(s+4)]$$

$$= e^{-4t} \mathcal{L}^{-1}\left(\frac{1}{s^2 + 9}\right)$$

$$= e^{-4t} \cdot \frac{1}{3} \cdot \sin 3t = \frac{e^{-4t} \sin 3t}{3} = f(t)$$

By second shifting theorem

$$\mathcal{L}^{-1}(e^{-as} \bar{f}(s)) = \begin{cases} f(t-a) & \text{if } t > a \\ 0 & \text{if } t < a \end{cases}$$

$$\mathcal{L}^{-1}\left(e^{-2s} \cdot \frac{1}{s^2 + 8s + 25}\right) = \frac{1}{3} e^{-4(t-2)} \sin 3(t-2) \quad \text{if } t > 2$$

$$= 0 \quad \text{if } t < 2$$

$$2). \quad \mathcal{L}^{-1} \left[\frac{e^{-3s}}{(s-4)^2} \right]$$

$$\begin{aligned} \text{Here } \bar{f}(s) &= \frac{1}{(s-4)^2}, \quad \mathcal{L}^{-1}[\bar{f}(s)] = \mathcal{L}^{-1} \left[\frac{1}{(s-4)^2} \right] \\ &= e^{4t} \mathcal{L}^{-1} \left[\frac{1}{s^2} \right] \\ &= \underline{e^{4t} t = f(t)} \end{aligned}$$

By second shifting theorem

$$\begin{aligned} \mathcal{L}^{-1} [e^{as} \bar{f}(s)] &= f(t-a), \quad \text{if } t > a \\ &= 0, \quad \text{if } t < a \end{aligned}$$

$$\begin{aligned} \text{i.e. } \mathcal{L}^{-1} \left[e^{3s} \frac{1}{(s-4)^2} \right] &= e^{4(t-3)} (t-3), \quad \text{if } t > 3 \\ &= 0, \quad \text{if } t < 3. \end{aligned}$$

$$\begin{aligned} 3) \quad \mathcal{L}^{-1} \left[\frac{e^{4-3s}}{(s+4)^{5/2}} \right] &= \mathcal{L}^{-1} \left[e^4 \frac{e^{-3s}}{(s+4)^{5/2}} \right] \\ &= e^4 \mathcal{L}^{-1} \left[e^{-3s} \frac{1}{(s+4)^{5/2}} \right] \end{aligned}$$

$$4) \quad \mathcal{L}^{-1} \left[e^{\pi s} \frac{1}{s^2 + 4} \right]$$

$$5) \quad \mathcal{L}^{-1} \left[e^{-2s} \frac{s}{s^2 + 3s + 2} \right]$$

Change of scale property:-

If $L^{-1}(\bar{f}(s)) = f(t)$, then $L^{-1}(\bar{f}(as)) = \frac{1}{a} f(t/a)$ if $t \rightarrow \infty$

Summary:-

1) If $L^{-1}\left(\frac{s}{(s^2+1)^2}\right) = \frac{t \sin t}{2}$ find $L^{-1}\left(\frac{8s}{(4s^2+1)^2}\right)$

Sol Given $L^{-1}\left(\frac{s}{(s^2+1)^2}\right) = \frac{t \sin t}{2} = f(t)$

$$\therefore L^{-1}\left(\frac{8s}{(4s^2+1)^2}\right) = L^{-1}\left(\frac{4(2s)}{((2s)^2+1)^2}\right)$$

$$= 4 \cdot L^{-1}\left(\frac{2s}{((2s)^2+1)^2}\right)$$

$$= 4 \times \frac{1}{2} \times \frac{t}{2} \sin(t/2)$$

$$= \frac{t}{2} \sin(t/2)$$

2) If $L^{-1}\left(\frac{s}{(s^2+1)^2}\right) = \frac{t \sin t}{2}$, Find $L^{-1}\left(\frac{32s}{(16s^2+1)^2}\right)$

Sol Given $L^{-1}\left(\frac{s}{(s^2+1)^2}\right) = \frac{t \sin t}{2} = f(t)$

$$\therefore \mathcal{L}^{-1}\left(\frac{32s}{(16s^2+1)^2}\right) = \mathcal{L}^{-1}\left(\frac{8(4s)}{(4s^2+1)^2}\right)$$

$$= 8 \cdot \left[\frac{1}{4} \times \frac{1}{2} \times \frac{1}{4} \sin\left(\frac{t}{4}\right) \right] = \frac{t}{4} \sin\left(\frac{t}{4}\right)$$

Inverse Laplace Transform of derivatives :-

$$\text{If } \mathcal{L}^{-1}(\bar{f}(s)) = f(t), \text{ then } \mathcal{L}^{-1}\left(\frac{d}{ds}(\bar{f}(s))\right) = -t \cdot f(t)$$

Sum

1) Find the inverse Laplace transform of $\log\left(\frac{s+1}{s-1}\right)$

$$\text{Sol} \quad \text{Let } \bar{f}(s) = \log\left(\frac{s+1}{s-1}\right)$$

$$= \log(s+1) - \log(s-1)$$

$$\mathcal{L}^{-1}\left(\frac{d}{ds}(\bar{f}(s))\right) =$$

$$\frac{d}{ds}(\bar{f}(s)) = \frac{1}{s+1} - \frac{1}{s-1}$$

$$\mathcal{L}^{-1}\left(\frac{d}{ds}(\bar{f}(s))\right) = \mathcal{L}^{-1}\left(\frac{1}{s+1}\right) - \mathcal{L}^{-1}\left(\frac{1}{s-1}\right)$$

$$= e^{-t} - e^t$$

$$\text{But } \mathcal{L}^{-1}\left(\frac{d}{ds}(\bar{f}(s))\right) = -t \cdot f(t)$$

$$-t \cdot f(t) = e^{-t} - e^t$$

$$f(t) = \frac{e^t - e^{-t}}{t}$$

$$\therefore \mathcal{L}^{-1}\left(\log\left(\frac{s+1}{s-1}\right)\right) = \frac{2}{t} \cdot \frac{e^t - e^{-t}}{2} = \frac{2 \sinh t}{t}$$

2) Find $\mathcal{L}^{-1}\left(\log\left(\frac{s^2-a^2}{s^2}\right)\right)$

sol let $\bar{f}(s) = \log\left(\frac{s^2-a^2}{s^2}\right)$
 $= \log(s^2-a^2) - \log(s^2)$

$$\frac{d}{ds}(\bar{f}(s)) = \frac{2s}{s^2-a^2} - \frac{2s}{s^2}$$

$$= \frac{2s}{s^2-a^2} - \frac{2}{s}$$

$$\mathcal{L}^{-1}\left(\frac{d}{ds}(\bar{f}(s))\right) = 2\mathcal{L}^{-1}\left(\frac{s}{s^2-a^2}\right) - 2\mathcal{L}^{-1}\left(\frac{1}{s}\right)$$

$$= 2\cosh at - 2$$

$$= 2(\cosh at - 1)$$

But $\mathcal{L}^{-1}\left(\frac{d}{ds}(\bar{f}(s))\right) = -t \cdot f(t)$

$$-t \cdot f(t) = 2(\cosh at - 1)$$

$$f(t) = \frac{2(1 - \cosh at)}{t}$$

$$\therefore \mathcal{L}^{-1}\left(\log\left(\frac{s^2-a^2}{s^2}\right)\right) = \frac{2}{t}(1 - \cosh at)$$

3) Find $\mathcal{L}^{-1}\left(\cot^{-1}(s/2)\right)$

sol let $\bar{f}(s) = \cot^{-1}(s/2)$

$$\frac{d}{ds}(\bar{f}(s)) = \frac{-1}{1+(s/2)^2} \cdot \frac{1}{2} = \frac{-1/2}{s^2/4 + 1} = \frac{-2}{s^2 + 4}$$

$$\mathcal{L}^{-1}\left(\frac{d}{ds}(\bar{f}(s))\right) = -\mathcal{L}^{-1}\left(\frac{2}{s^2+2^2}\right) \\ = -\sin 2t$$

$$\text{But } \mathcal{L}^{-1}\left(\frac{d}{ds}(\bar{f}(s))\right) = -t \cdot f(t)$$

$$-t \cdot f(t) = -\sin 2t$$

$$f(t) = \frac{\sin 2t}{t}$$

$$\therefore \mathcal{L}^{-1}\left(\cot^{-1}(s/2)\right) = \frac{\sin 2t}{t}$$

$$3) \text{ Find } \mathcal{L}^{-1}\left(\tan^{-1}(2/s)\right)$$

$$\text{eg let } \bar{f}(s) = \tan^{-1}(2/s)$$

$$\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$$

$$\frac{d}{ds}(\bar{f}(s)) = \frac{1}{1+(2/s)^2} \times 2 \times \left(-\frac{1}{s^2}\right)$$

$$= \frac{s^2}{s^2+4} \times \frac{-2}{s^2}$$

$$= -\frac{2}{s^2+4}$$

$$\mathcal{L}^{-1}\left(\frac{d}{ds}(\bar{f}(s))\right) = -\mathcal{L}^{-1}\left(\frac{2}{s^2+2^2}\right)$$

$$= -\sin 2t$$

$$\therefore \mathcal{L}^{-1}\left(\frac{d}{ds}(\bar{f}(s))\right) = -t \cdot f(t)$$

$$-t \cdot f(t) = -\sin 2t$$

$$f(t) = \frac{\sin 2t}{t}$$

$$\therefore \mathcal{L}^{-1}\left(\tan^{-1}(2/s)\right) = \frac{\sin 2t}{t}$$

Inverse Laplace Transform of Integrals:

$$\text{If } L^{-1}\{\bar{f}(s)\} = f(t) \text{ then } L^{-1}\left\{\int_s^{\infty} \bar{f}(s) ds\right\} = \frac{f(t)}{t}$$

$$f(t) = t \cdot L^{-1}\left\{\int_s^{\infty} \bar{f}(s) ds\right\}$$

Sum

1) Find $L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}$

so let $\bar{f}(s) = \frac{s}{(s^2+a^2)^2}$

By the application of inverse L.T of Integrals

$$f(t) = t \cdot L^{-1}\left\{\int_s^{\infty} \bar{f}(s) ds\right\}$$

$$= t \cdot L^{-1}\left\{\int_s^{\infty} \frac{s}{(s^2+a^2)^2} \cdot ds\right\}$$

$$= t \cdot \frac{1}{2} L^{-1}\left\{\int_s^{\infty} \frac{2s}{(s^2+a^2)^2} \cdot ds\right\}$$

$$= t \cdot \frac{1}{2} L^{-1}\left\{\left[\frac{(s^2+a^2)^{-2+1}}{-2+1}\right]_s^{\infty}\right\}$$

$$= \frac{t}{2} L^{-1}\left\{\left[-\frac{1}{s^2+a^2}\right]_s^{\infty}\right\}$$

$$= \frac{t}{2} L^{-1}\left\{-\left(0 - \frac{1}{s^2+a^2}\right)\right\}$$

$$= \frac{t}{2} L^{-1}\left\{\frac{1}{s^2+a^2}\right\}$$

$$= \frac{t}{2a} \sin at$$

$$\boxed{\int (f(x))^n (f'(x)) dx = \frac{(f(x))^{n+1}}{n+1} + C}$$

2) Find $L^{-1}\left(\frac{s+2}{(s^2+4s+8)^2}\right)$

sol] let $\bar{f}(s) = \frac{s+2}{(s^2+4s+8)^2}$

~~$= \frac{s+2}{((s+2)^2+4)^2}$~~

~~$= \frac{s+2}{((s+2)^2+2^2)^2}$~~

$s^2+2 \cdot s \cdot 2 + 2^2 + 4$
 $((s+2)^2+4)^2$

By the application of L.T of Integrals

$f(t) = t \cdot L^{-1}\left[\int_s^{\infty} \frac{s+2}{(s^2+4s+8)^2} ds\right]$

$= t \cdot \frac{1}{2} L^{-1}\left[\int_s^{\infty} \frac{2s+4}{(s^2+4s+8)^2} ds\right]$

$= \frac{t}{2} \cdot L^{-1}\left[\int_s^{\infty} (s^2+4s+8)^{-2} (2s+4) ds\right]$

$= \frac{t}{2} L^{-1}\left[\frac{(s^2+4s+8)^{-2+1}}{-2+1}\right]_s^{\infty} \left[\int f(s) ds\right]^n$

$= \frac{t}{2} L^{-1}\left[\frac{-1}{(s^2+4s+8)^1}\right]_s^{\infty}$

$= \frac{t}{2} L^{-1}\left(0 - \frac{1}{s^2+4s+8}\right)$

$= \frac{t}{2} L^{-1}\left(\frac{1}{s^2+4s+8}\right) = \frac{t}{2} L^{-1}\left(\frac{1}{(s+2)^2+2^2}\right)$

$= \frac{t}{2} L^{-1}(\bar{f}(s+2)) = \frac{t}{2} e^{-2t} \cdot L^{-1}(\bar{f}(s))$

$$= \frac{t}{2} e^{-2t} \mathcal{L}^{-1} \left(\frac{1}{s^2 + 2} \right)$$

$$= \frac{t}{4} e^{-2t} \mathcal{L}^{-1} \left(\frac{2}{s^2 + 2} \right) = \frac{t}{4} e^{-2t} \sin 2t$$

3) Find $\mathcal{L}^{-1} \left(\int_s^\infty \log \left(\frac{s+b}{s+a} \right) ds \right)$

sol let $\bar{f}(s) = \log \left(\frac{s+b}{s+a} \right)$

$$= \log(s+b) - \log(s+a)$$

$$\frac{d}{ds} (\bar{f}(s)) = \cancel{\log} \left(\cancel{\log} \right) = \frac{1}{s+b} - \frac{1}{s+a}$$

$$\mathcal{L}^{-1} \left(\frac{d}{ds} (\bar{f}(s)) \right) = \mathcal{L}^{-1} \left(\frac{1}{s+b} \right) - \mathcal{L}^{-1} \left(\frac{1}{s+a} \right)$$

$$= e^{-bt} - e^{-at}$$

$$-t \cdot f(t) = e^{-bt} - e^{-at}$$

$$f(t) = \frac{e^{-at} - e^{-bt}}{t}$$

By application

Given $\mathcal{L}^{-1} \left(\int_s^\infty \log \left(\frac{s+b}{s+a} \right) ds \right) = \frac{f(t)}{t}$

$$= \frac{e^{-at} - e^{-bt}}{t \times t} = \frac{e^{-at} - e^{-bt}}{t^2}$$

4) Find $\mathcal{L}^{-1} \left(\int_s^\infty \log \left(\frac{u-1}{u+1} \right) du \right)$

sol let $\bar{f}(u) = \log \left(\frac{u-1}{u+1} \right)$

$$= \log(u-1) - \log(u+1)$$

$$\frac{d}{du}(\bar{f}(u)) = \frac{1}{u-1} - \frac{1}{u+1}$$

$$\mathcal{L}^{-1} \left(\frac{d}{du}(\bar{f}(u)) \right) = \mathcal{L}^{-1} \left(\frac{1}{u-1} - \frac{1}{u+1} \right)$$

$$= e^t - e^{-t}$$

But $-t \cdot f(t) = e^t - e^{-t}$

$$f(t) = \frac{e^{-t} - e^t}{t}$$

$$= 0$$

$$= -\frac{2}{t} \sinh t$$

$$\therefore \mathcal{L}^{-1} \left(\int_s^\infty \log \left(\frac{u-1}{u+1} \right) du \right) = -\frac{2}{t^2} \sinh t \quad \text{i.e., } \frac{f(t)}{t}$$

$$\begin{aligned}
 &= e^{2t} \mathcal{L}^{-1} \left[\frac{3s+4}{s^2+16} \right] \\
 &= e^{2t} \left[\mathcal{L}^{-1} \left[\frac{3s}{s^2+4^2} \right] + \mathcal{L}^{-1} \left[\frac{4}{s^2+4^2} \right] \right] \\
 &= e^{2t} \left[3 \mathcal{L}^{-1} \left[\frac{s}{s^2+4^2} \right] + \mathcal{L}^{-1} \left[\frac{4}{s^2+4^2} \right] \right] \\
 &= e^{2t} (3 \cos 4t + \sin 4t)
 \end{aligned}$$

4. Find $\mathcal{L}^{-1} \left[\frac{2s+12}{s^2+6s+13} \right]$

$$\begin{aligned}
 \mathcal{L}^{-1} \left[\frac{2s+12}{s^2+6s+13} \right] &= \mathcal{L}^{-1} \left[\frac{2s+12}{s^2+2(3)s+9+4} \right] = \mathcal{L}^{-1} \left[\frac{2s+12}{(s+3)^2+4} \right] \\
 &= \mathcal{L}^{-1} \left[\frac{2s+6+6}{(s+3)^2+4} \right] = \mathcal{L}^{-1} \left[\frac{2(s+3)+6}{(s+3)^2+4} \right] \\
 &= \mathcal{L}^{-1} \left[\frac{f(s+3)}{(s+3)^2+4} \right] = e^{-3t} \mathcal{L}^{-1} \left[\frac{f(s)}{s^2+4} \right] \\
 &= e^{-3t} \mathcal{L}^{-1} \left[\frac{2s+6}{s^2+4} \right] = e^{-3t} \left[2 \mathcal{L}^{-1} \left[\frac{s}{s^2+4} \right] + 6 \mathcal{L}^{-1} \left[\frac{1}{s^2+4} \right] \right] \\
 &= e^{-3t} \left[2 \mathcal{L}^{-1} \left[\frac{s}{s^2+2^2} \right] + 6 \right] = e^{-3t} \left[2 \cos 2t + 6 \left(\frac{1}{2} \sin 2t \right) \right] \\
 \mathcal{L}^{-1} \left[\frac{2s+12}{s^2+6s+13} \right] &= e^{-3t} \left[2 \cos 2t + 3 \sin 2t \right]
 \end{aligned}$$

* Convolution:

Def:- Let $f(t)$ and $g(t)$ be two functions defined for $t > 0$ then convolution product of $f(t)$ and $g(t)$ is defined as $f(t) * g(t) = \int_0^t f(u)g(t-u)du$

properties:-

(i) Convolution product is commutative.

$$f(t) * g(t) = g(t) * f(t)$$

(ii) Convolution product is Associative

* Convolution:

Def:- Let $f(t)$ and $g(t)$ be two functions defined for $t > 0$ then convolution product of $f(t)$ and $g(t)$ is defined as $f(t) * g(t) = \int_0^t f(u)g(t-u)du$

properties:-

(i) Convolution product is commutative.

$$f(t) * g(t) = g(t) * f(t)$$

(ii) Convolution product is Associative

$$f(t) * [g(t) * h(t)] = [f(t) * g(t)] * h(t)$$

(iii) $f(t) * 0 = 0 * f(t) = 0$.

Convolution theorem:-

If $L[f(t)] = \bar{f}(s)$, $L[g(t)] = \bar{g}(s)$ then

$$L[f(t) * g(t)] = \bar{f}(s) \cdot \bar{g}(s)$$

$$\text{or) } L^{-1}[\bar{f}(s) \bar{g}(s)] = f(t) * g(t) = \int_0^t f(u) g(t-u) du$$

1. Using Convolution theorem, find $L^{-1}\left[\frac{s^2}{(s^2+4)(s^2+25)}\right]$

$$\text{Given } L^{-1}\left[\frac{s^2}{(s^2+4)(s^2+25)}\right] = L^{-1}\left[\frac{s}{s^2+4} \cdot \frac{s}{s^2+25}\right]$$

$$\text{Let } \bar{f}(s) = \frac{s}{s^2+4} \text{ and } \bar{g}(s) = \frac{s}{s^2+25}$$

$$\text{then } L^{-1}[\bar{f}(s)] = L^{-1}\left[\frac{s}{s^2+4}\right] = \cos 2t = f(t) \text{ and}$$

$$L^{-1}[\bar{g}(s)] = L^{-1}\left[\frac{s}{s^2+25}\right] = \cos 5t = g(t)$$

$$\therefore \text{By Convolution theorem, } L^{-1}[\bar{f}(s) \cdot \bar{g}(s)] = f(t) * g(t) = \int_0^t f(u) g(t-u) du$$

$$= \int_0^t \cos 2u \cos 5(t-u) du$$

$$= \int_0^t \cos 2u \cos (5t - 5u) du$$

$$= \frac{1}{2} \int_0^t 2 \cos 2u \cos (5t - 5u) du$$

$$= \frac{1}{2} \int_0^t [\cos (5t - 3u) + \cos (7u - 5t)] du$$

$$= \frac{1}{2} \left[\int_0^t \cos (3u - 5t) du + \int_0^t \cos (7u - 5t) du \right] \quad [\because \cos(-\theta) = \cos \theta]$$

$$= \frac{1}{2} \left[\frac{\sin 3u - 5t}{3} + \frac{\sin (7u - 5t)}{7} \right]_0^t$$

$$= \frac{1}{2} \left[\left(\frac{\sin (-2t)}{3} + \frac{\sin (2t)}{7} \right) - \left(\frac{\sin (-5t)}{3} + \frac{\sin (-5t)}{7} \right) \right]$$

$$= \frac{1}{2} \left[-\frac{\sin 2t}{3} + \frac{\sin 2t}{7} + \frac{\sin 5t}{3} + \frac{\sin 5t}{7} \right]$$

$$[\because \sin(-\theta) = -\sin \theta]$$

$$= \frac{1}{2} \left[\frac{-7 \sin 2t + 3 \sin 2t - 7 \sin t + 3 \sin t}{21} \right]$$

$$= \frac{1}{2} \left[\frac{10 \sin t - 4 \sin 2t}{21} \right] = \frac{1}{21} [5 \sin t - 2 \sin 2t]$$

2. Using convolution theorem find $\mathcal{L}^{-1} \left[\frac{1}{(s-2)(s+2)^2} \right]$

$$\text{Given } \mathcal{L}^{-1} \left[\frac{1}{(s-2)(s+2)^2} \right] = \mathcal{L}^{-1} \left[\frac{1}{(s-2)(s+2)(s+2)} \right] = \mathcal{L}^{-1} \left[\frac{1}{(s-2)} \cdot \frac{1}{(s+2)^2} \right]$$

$$\text{Let } \bar{f}(s) = \frac{1}{s-2} \quad \& \quad \bar{g}(s) = \frac{1}{(s+2)^2}$$

$$\text{Then } \mathcal{L}^{-1} [\bar{f}(s)] = \mathcal{L}^{-1} \left[\frac{1}{s-2} \right] = e^{2t} = f(t)$$

$$\text{and } \mathcal{L}^{-1} [\bar{g}(s)] = \mathcal{L}^{-1} \left[\frac{1}{(s+2)^2} \right] = e^{-2t} \mathcal{L}^{-1} [\bar{f}(s+2)] = e^{-2t} \mathcal{L}^{-1} \left[\frac{1}{s} \right] \\ = e^{-2t} \mathcal{L}^{-1} \left[\frac{1}{s} \right] = e^{-2t} t = g(t)$$

$$\text{By C.T } \mathcal{L}^{-1} [\bar{f}(s) \cdot \bar{g}(s)] = f(t) * g(t) = \int_0^t f(u) g(t-u) du$$

$$= \int_0^t e^{2u} e^{-2(t-u)} (t-u) du \\ = \int_0^t e^{2u} e^{-2t+2u} (t-u) du$$

$$= \int_0^t e^{2u-2t+2u} (t-u) du \\ = e^{-2t} \int_0^t e^{4u} (t-u) du$$

$$= e^{-2t} \left[(t-u) \frac{e^{4u}}{4} - \int (-1) \frac{e^{4u}}{4} du \right]$$

$$= e^{-2t} \left[\left(0 - \frac{te^{4(0)}}{4} \right) + \left(\frac{e^{4u}}{16} \right) \Big|_0^t \right]$$

$$= e^{-2t} \left[-\frac{t}{4} + \frac{e^{4t}}{16} - \frac{e^0}{16} \right]$$

$$= e^{-2t} \left[\frac{-4t + e^{4t} - 1}{16} \right] = \frac{e^{-2t}}{16} (e^{4t} - (4t+1))$$

3. Using Convolution Theorem find $\mathcal{L}^{-1} \left(\frac{1}{(s^2+4)(s+1)} \right)$.

Given $\mathcal{L}^{-1} \left(\frac{1}{(s^2+4)(s+1)} \right) = \mathcal{L}^{-1} \left(\frac{1}{(s^2+4)(s+1)} \cdot \frac{1}{s+1} \right)$

Let $\bar{f}(s) = \frac{1}{(s^2+4)(s+1)} = \frac{As+B}{s^2+4} + \frac{C}{s+1}$

Then $1 = (As+B)(s+1) + C(s^2+4)$

$1 = 0 + 5C \Rightarrow C = 1/5$

Comparing Coefficients of s^2 & s we get

$0 = A+C \Rightarrow A = -C = -1/5$

$0 = A+B \Rightarrow B = -A = -(-1/5) = 1/5$

$\therefore \bar{f}(s) = \frac{-1/5 s + 1/5}{s^2+4} + \frac{1/5}{s+1}$

$\therefore \mathcal{L}^{-1}[\bar{f}(s)] = \mathcal{L}^{-1} \left[\frac{-1/5 s}{s^2+4} + \frac{1/5}{s^2+4} + \frac{1/5}{s+1} \right]$

$= -\frac{1}{5} \mathcal{L}^{-1} \left(\frac{s}{s^2+2^2} \right) + \frac{1}{5} \mathcal{L}^{-1} \left(\frac{1}{s^2+4} \right) + \frac{1}{5} \mathcal{L}^{-1} \left(\frac{1}{s+1} \right)$

$= -\frac{1}{5} \cos 2t + \frac{1}{5 \cdot 2} \mathcal{L}^{-1} \left(\frac{2}{s^2+2^2} \right) + \frac{1}{5} e^{-t}$

$= -\frac{1}{5} \cos 2t + \frac{1}{10} \sin 2t + \frac{1}{5} e^{-t} = f(t)$

and $\bar{g}(s) = \frac{1}{s+1}$ then $\mathcal{L}^{-1}(\bar{g}(s)) = \mathcal{L}^{-1} \frac{1}{s+1} = e^{-t} = g(t)$

By Convolution Theorem $\mathcal{L}^{-1}[\bar{f}(s) \cdot \bar{g}(s)] = \int_0^t f(u) g(t-u) du$

$= \int_0^t \left[-\frac{1}{5} \cos 2u + \frac{1}{10} \sin 2u + \frac{1}{5} e^{-u} \right] e^{-(t-u)} du$

$= \int_0^t \left[-\frac{1}{5} \cos 2u + \frac{1}{10} \sin 2u + \frac{1}{5} e^{-u} \right] e^{-t} e^u du$

$= e^{-t} \int_0^t \left[-\frac{1}{5} \cos 2u + \frac{1}{10} \sin 2u + \frac{1}{5} e^{-u} \right] e^u du$

$= e^{-t} \left[-\frac{1}{5} \int_0^t \cos 2u du + \frac{1}{10} \int_0^t \sin 2u du + \frac{1}{5} \int_0^t e^{-u} du \right]$

$$\begin{aligned}
&= e^{-t} \left[\frac{-1}{5} \left(\frac{e^u}{1+2^u} (\cos 2u + 2 \sin 2u) \right) \Big|_0^t + \frac{1}{10} \left(\frac{e^u}{1+2^u} (\sin 2u + 2 \cos 2u) \right) \Big|_0^t \right. \\
&\quad \left. + \frac{1}{5} (u) \Big|_0^t \right] \\
&\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2} (a \sin bx - b \cos bx) \quad \int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2} (a \cos bx + b \sin bx) \\
&= e^{-t} \left[\left(\frac{-1}{25} \right) (e^t (\cos 2t + 2 \sin 2t) - e^0 (\cos 0 + 2 \sin 0)) \right] + \frac{1}{50} \\
&\quad \left[e^t (\sin 2t - 2 \cos 2t) - e^0 (\sin 0 - 2 \cos 0) \right] + \frac{1}{5} (t-0) \\
&= e^{-t} \left[-\frac{1}{25} (e^t \cos 2t + 2 \sin 2t - 1) + \frac{1}{50} (e^t \sin 2t - 2e^t \cos 2t \right. \\
&\quad \left. + 2) + \frac{1}{5} t \right] \\
&= \frac{1}{25} \left[-e^t e^t \cos 2t + 2e^t e^t \sin 2t - e^t \right] + \frac{1}{50} \left[e^t e^t \sin 2t \right. \\
&\quad \left. - 2e^t e^t \cos 2t + 2e^t \right] + \frac{1}{5} t e^{-t} \\
&= -\frac{1}{25} [\cos 2t + 2 \sin 2t - e^{-t}] + \frac{1}{50} [\sin 2t - 2 \cos 2t + 2e^{-t}] + \frac{1}{5} t e^{-t} \\
&= \frac{-2 \cos 2t - 4 \sin 2t + 2e^{-t} + \sin 2t - 2 \cos 2t + 2e^{-t} + 10 t e^{-t}}{50} \\
&= \frac{1}{50} (10 t e^{-t} + 2e^{-t} - 4 \cos 2t - 3 \sin 2t)
\end{aligned}$$

4) Using L.T. solve the equation

$$y(t) = 1 - e^{-t} + \int_0^t y(t-u) \sin u \, du$$

Sol:- By def of convolution $f(t) * g(t) = \int_0^t f(u) g(t-u) \, du$

$$y(t) = 1 - e^{-t} + \sin t * y(t) \quad \left[\begin{array}{l} \text{Here } f(u) = \sin u \\ g(t-u) = y(t-u) \end{array} \right]$$

$$\text{i.e. } y(t) = 1 - e^{-t} + y(t) * \sin t$$

$$\begin{aligned} L[y(t)] &= L[1 - e^{-t} + y(t) * \sin t] \\ &= L(1) - L(e^{-t}) + L[y(t) * \sin t] \\ &= \frac{1}{s} - \frac{1}{s+1} + L[y(t)] L(\sin t) \\ &= \frac{s+1-s}{(s+1)s} + L[y(t)] \frac{1}{s^2+1} \end{aligned}$$

$$L[y(t)] = \frac{1}{s(s+1)} + \frac{L[y(t)]}{s^2+1}$$

$$\Rightarrow L[y(t)] - \frac{L[y(t)]}{s^2+1} = \frac{1}{(s+1)s}$$

$$\Rightarrow \left[1 - \frac{1}{s^2+1} \right] L[y(t)] = \frac{1}{(s+1)s}$$

$$\Rightarrow \frac{(s^2+1-1)}{s^2+1} L[y(t)] = \frac{1}{(s+1)s}$$

$$\Rightarrow L[y(t)] = \frac{s^2+1}{s^3(s+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{s+1}$$

After finding the ~~solution~~ constants,

$$\boxed{y(t) = 1 - \frac{t}{2} + \frac{t^2}{2} - 2e^{-t} - t + \frac{t^2}{2}} \quad \checkmark$$

$$A = 2, B = -1, C = 1, D = 1$$

(Differential Equations)

* Solution of d.e by Laplace Transforms:-

Procedure

Step 1:- Apply Laplace Transform on both sides of given differential equation.

Step 2:- Simplify the equation using Laplace transform of derivatives and other methods.

Step 3:- Substitute the initial conditions.

Step 4:- Simplify the equation to the form of $\bar{y}(s)$.

Step 5:- Find $L^{-1}[\bar{y}(s)]$ to get the solution $y(t)$.

1. Solve the d.e $\frac{d^3 y}{dt^3} + 2\frac{d^2 y}{dt^2} - \frac{dy}{dt} - 2y = 0$.

$$y(0) = 1, y'(0) = y''(0) = 2.$$

Sol:- Given d.e can also be written as $y''' + 2y'' - y' - 2y = 0$

$$y'''(t) + 2y''(t) - y'(t) - 2y = 0 \quad \text{--- (1)}$$

Taking Laplace Transforms, on both sides we get

$$L[y'''(t) + 2y''(t) - y'(t) - 2y] = L[0]$$

$$L[y'''(t)] + 2L[y''(t)] - L[y'(t)] - 2L[y] = 0$$

$$[s^3 \bar{y}(s) - s^2 y(0) - s y'(0) - y''(0)] + 2[s^2 \bar{y}(s) - s y(0) - y'(0)] - [s \bar{y}(s) - y(0)] - 2\bar{y}(s) = 0$$

$$[s^3 \bar{y}(s) - s^2(1) - s(2) - 2] + 2[s^2 \bar{y}(s) - s(1) - 2] - [s \bar{y}(s) - 1] - 2\bar{y}(s) = 0$$

Using given Conditions $y(0) = 1, y'(0) = y''(0) = 2$.

$$[s^3 \bar{y}(s) - s^2(1) - s(2) - 2] + 2[s^2 \bar{y}(s) - s(1) - 2] - [s \bar{y}(s) - 1] - 2\bar{y}(s) = 0$$

$$\bar{y}(s)[s^3 + 2s^2 - s - 2] - s^2 - 2s - 2 - 2s - 4 + 1 = 0$$

$$\bar{y}(s)(s^3 + 2s^2 - s - 2) - s^2 - 4s - 5 = 0$$

$$\Rightarrow (s^3 + 2s^2 - s - 2) \bar{y}(s) = s^2 + 4s + 5$$

$$\bar{y}(s) = \frac{s^2 + 4s + 5}{s^3 + 2s^2 - s - 2}$$

$$\bar{y}(s) = \frac{s^2 + 4s + 5}{s^2(s+2) - 1(s+2)} = \frac{s^2 + 4s + 5}{(s^2 - 1)(s+2)}$$

$$\bar{y}(s) = \frac{s^2 + 4s + 5}{(s+2)(s+1)(s-1)} = \frac{A}{(s+2)} + \frac{B}{s+1} + \frac{C}{s-1}$$

$$s^2 + 4s + 5 = A(s+1)(s-1) + B(s+2)(s-1) + C(s+2)(s+1) \quad \text{--- (2)}$$

Put $s = 1$ in (2) we get

$$10 = 6C \Rightarrow C = \frac{5}{3}$$

put $s = -1$ in (2) we get $2 = -2B \Rightarrow B = -1$

put $s = -2$ in (2) we get $1 = 3A \Rightarrow A = 1/3$

$$\therefore \bar{y}(s) = \frac{1/3}{s+2} - \frac{1}{s+1} + \frac{s/3}{s-1}$$

$$\mathcal{L}^{-1}[\bar{y}(s)] = \mathcal{L}^{-1}\left[\frac{1/3}{s+2} - \frac{1}{s+1} + \frac{s/3}{s-1}\right]$$

$$= \frac{1}{3} \mathcal{L}^{-1}\left[\frac{1}{s+2}\right] - \mathcal{L}^{-1}\left[\frac{1}{s+1}\right] + \frac{s}{3} \mathcal{L}^{-1}\left[\frac{1}{s-1}\right]$$

$$= \frac{1}{3} e^{-2t} - e^{-t} + \frac{s}{3} e^t = y(t)$$

2. Solve the d.e $\frac{dx}{dt} - 4\frac{dx}{dt} - 12x = e^{3t}$, $x(0) = 1, x'(0) = -2$

Given d.e can be written as $x''(t) - 4x'(t) - 12x(t) = e^{3t}$ (1)

Taking Laplace Transform on both sides we get

$$\mathcal{L}[x''(t) - 4x'(t) - 12x(t)] = \mathcal{L}[e^{3t}]$$

$$\mathcal{L}[x''(t)] - 4\mathcal{L}[x'(t)] - 12\mathcal{L}[x(t)] = \frac{1}{s-3}$$

$$[s^2 \bar{x}(s) - sx(0) - x'(0)] - 4[s\bar{x}(s) - x(0)] - 12\bar{x}(s) = \frac{1}{s-3}$$
$$\bar{x}(s)[s^2 - 4s - 12] = \frac{1}{s-3}$$

Using given conditions $x(0) = 1, x'(0) = -2$.

$$[s^2 \bar{x}(s) - s(1) - (-2)] - 4[s\bar{x}(s) - 1] - 12\bar{x}(s) = \frac{1}{s-3}$$

$$(s^2 - 4s - 12)\bar{x}(s) - s + 2 + 4 = \frac{1}{s-3}$$

$$(s^2 - 4s - 12)\bar{x}(s) - s + 6 = \frac{1}{s-3}$$

$$(s^2 - 4s - 12)\bar{x}(s) = \frac{1}{s-3} + s - 6$$

$$= \frac{1 + (s-6)(s-3)}{(s-3)}$$

$$= \frac{1 + s^2 - 9s + 18}{(s-3)}$$

$$= \frac{s^2 - 9s + 19}{(s-3)}$$

$$\bar{x}(s) = \frac{s^2 - 9s + 19}{(s-3)(s^2 - 4s - 12)} = \frac{s^2 - 9s + 19}{(s-3)(s^2 - 6s + 2s - 12)}$$

$$= \frac{s^2 - 9s + 19}{(s-3)(s(s-6) + 2(s-6))}$$

$$\bar{x}(s) = \frac{s^2 - 9s + 19}{(s-3)((s-6)(s+2))}$$

$$\frac{s^2 - 9s + 19}{(s-3)(s-6)(s+2)} = \frac{A}{s-3} + \frac{B}{s-6} + \frac{C}{s+2} \quad \text{--- (2)}$$

$$s^2 - 9s + 19 = A(s-6)(s+2) + B(s-3)(s+2) + C(s-3)(s-6) \quad \text{--- (2)}$$

put $s=3$ in (2) we get

$$9 - 27 + 19 = A(3-6)(3+2) + 0 + 0 \Rightarrow 1 = -15A \Rightarrow A = -1/15$$

put $s=6$ in (2) we get

$$36 - 54 + 19 = 0 + B(6-3)(6+2) + 0 \Rightarrow 1 = 24B \Rightarrow B = 1/24$$

put $s=-2$ in (2) we get

$$4 + 18 + 19 = 0 + 0 + C(-2-3)(-2-6) \Rightarrow 41 = 40C \Rightarrow C = 41/40$$

$$\therefore \bar{x}(s) = \frac{-1/15}{s-3} + \frac{1/24}{s-6} + \frac{41/40}{s+2}$$

$$\mathcal{L}^{-1}(\bar{x}(s)) = \frac{-1}{15} \mathcal{L}^{-1}\left(\frac{1}{s-3}\right) + \frac{1}{24} \mathcal{L}^{-1}\left(\frac{1}{s-6}\right) + \frac{41}{40} \mathcal{L}^{-1}\left(\frac{1}{s+2}\right)$$

$$= \frac{-1}{15} e^{3t} + \frac{1}{24} e^{6t} + \frac{41}{40} e^{-2t}$$

$$= x(t)$$

* Solve the d.e $\frac{d^3x}{dt^3} + 9x = \sin t$, $x(0) = 1, x(\pi/2) = 1$

Given d.e can be written as $x''(t) + 9x(t) = \sin t$

Taking Laplace Transform on both sides we get

$$L[x''(t) + 9x(t)] = L[\sin t]$$

$$L[x''(t)] + 9L[x(t)] = L[\sin t]$$

$$s^3 \bar{x}(s) - s^2 x(0) - x'(0) + 9\bar{x}(s) = \frac{1}{s^2 + 1}$$

$$s^3 \bar{x}(s) - s - x'(0) + 9\bar{x}(s) = \frac{1}{s^2 + 1}$$

$$\bar{x}(s)[s^3 + 9] - s - x'(0) = \frac{1}{s^2 + 1}$$

Since $x'(0)$ is not given let it be $k \Rightarrow x'(0) = k$.

$$\bar{x}(s)[s^3 + 9] - s - k = \frac{1}{s^2 + 1}$$

$$(s^3 + 9)(\bar{x}(s)) = \frac{1}{s^2 + 1} + s + k$$

$$\bar{x}(s) = \frac{1}{(s^2 + 1)(s^2 + 9)} + \frac{s + k}{s^2 + 9}$$

$$= \frac{1}{8} \left[\frac{1}{s^2 + 1} - \frac{1}{s^2 + 9} \right] + \frac{s + k}{s^2 + 9}$$

$$= \frac{1}{8} \left(\frac{1}{s^2 + 1} \right) - \frac{1}{8} \left(\frac{1}{s^2 + 9} \right) + \frac{s}{s^2 + 9} + \frac{k}{s^2 + 9}$$

$$L^{-1}[\bar{x}(s)] = \frac{1}{8} L^{-1} \left(\frac{1}{s^2 + 1} \right) - \frac{1}{8} L^{-1} \left(\frac{1}{s^2 + 9} \right) + L^{-1} \left(\frac{s}{s^2 + 9} \right) + k L^{-1} \left(\frac{1}{s^2 + 9} \right)$$

$$x(t) = \frac{1}{8} \sin t - \frac{1}{8} \frac{1}{3} \sin 3t + \cos 3t + \frac{k}{3} \sin 3t$$

Ex 1 $x(t)$ = Given that $x(\pi/2) = 1$ i.e., $x = 1$ when $t = \pi/2$

$$1 = \frac{1}{8} \sin \pi/2 - \frac{1}{24} \sin 3\pi/2 + \cos 3\pi/2 + \frac{k}{3} \sin 3\pi/2$$

$$1 = \frac{1}{8}(1) - \frac{1}{24}(-1) + 0 + \frac{k}{3}(-1)$$

$$1 = \frac{1}{8} + \frac{1}{24} - \frac{k}{3}$$

$$\Rightarrow \frac{4}{24} - \frac{k}{3} = 1$$

$$\frac{k}{3} = \frac{1}{6} - 1 = -\frac{5}{6}$$

$$k = -15/6 = -5/2.$$

$$x(t) = \frac{1}{8} \sin t - \frac{1}{24} \sin 3t + \cos 3t - \frac{5}{6} \sin 3t$$

$$x(t) = \frac{3 \sin t + 24 \cos 3t - 21 \sin 3t}{24}$$

Apply convolution theorem to evaluate

$$1) \mathcal{L}^{-1} \left[\frac{s}{(s^2+a^2)^2} \right] \quad 2) \mathcal{L}^{-1} \left[\frac{s^2}{(s^2+a^2)(s^2+b^2)} \right]$$

$$3) \mathcal{L}^{-1} \left[\frac{1}{s(s+1)(s+2)} \right] \quad 4) \mathcal{L}^{-1} \left[\frac{s}{(s+4)^2(s^2-9)} \right]$$

1) Solve the d.e. $y''' - 3y'' + 3y' - y = t^2 e^t$, $y(0) = 1$,
 $y'(0) = 0$, $y''(0) = -2$.

2) $y'' - 3y' + 2y = 4t + e^{3t}$, $y(0) = 1$, $y'(0) = -1$

3) $\frac{d^2 y}{dt^2} + \frac{dy}{dt} - 2y = 3 \cos 3t - 11 \sin 3t$, $y(0) = 0$, $y'(0) = 6$